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The Nonlinear Stability of a Free Shear Layer in the Viscous Critical Layer Regime

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THE NONLINEAR STABILITY OF A FREE SHEAR LAYER IN THE VISCOUS CRITICAL LAYER RÉGIME

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CONTENTS

The nonlinear evolution of weakly amplified waves in a hyperbolic tangent free shear layer is described for spatially and temporally growing waves when the shear layer Reynolds number is large and the critical layer viscous.

An artificial body force is introduced in order to keep the mean flow parallel. Jump conditions on the perturbation velocity and mean vorticity are derived across the critical layer by applying the method of matched asymptotic expansions and it is shown that viscous effects outside the critical layer have to be taken into account in order to obtain a uniformly valid solution. Consequently the true neutral wavenumber and frequency are lower than their inviscid counterparts. When only the harmonic fluctuations are considered, it is known that the Landau constant is negative so that linearly amplified disturbances reach an equilibrium amplitude. It is shown that when the mean flow distortion generated by Reynolds stresses is also included, the Landau constant becomes positive. Thus, in both the spatial and temporal case, linearly amplified waves are further destabilized and damped waves are unstable above a threshold amplitude.

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1. Introduction

When a free shear layer is excited by a wave like disturbance of suitable frequency or wavenumber, the amplitude of the wave increases exponentially according to well-established linear stability considerations. This exponential increase clearly restricts the domain of validity of linear theory to very short times and distances. In order to overcome this difficulty, Stuart (1960) and Watson (1960) proposed a systematic expansion scheme whereby the nonlinear stability of parallel shear flows could be investigated for weakly amplified waves. This theory has been particularly successful in describing, for instance, the early evolution of Taylor vortices in the flow between concentric, rotating circular cylinders. It has also been applied to a variety of situations (plane Poiseuille flow, boundary layers . . .) where transition from laminar to turbulent flow takes place. The aim of this paper is to consider the particular case of a parallel free shear layer with a hyperbolic tangent velocity profile at high Reynolds numbers.

In carrying out such a study, it is hoped to gain further insight not only in the transition process of free shear layers, as studied experimentally by Freymuth (1966) and Miksad (1972), but also in the mechanisms governing the development of orderly structures in fully turbulent flows. As far as transition is concerned, Miksad (1972) showed how nonlinear effects, characterized by the appearance of harmonics and subharmonics, become important as soon as the amplitude of the fluctuations reaches 2 % of the maximum mean velocity. Even though Stuart's work cannot be expected to account for the appearance of subharmonic fluctuations, it provided a qualitatively satisfactory description of the finite amplitude equilibrium process of the fundamental disturbance. In the case of large-scale structures in turbulent flows, the jet-forcing experiments of Crow & Champagne (1971) and Moore (1977) have clearly shown that the exciting disturbances are amplified by the jet as if they were traveling in a fictitious laminar flow with the same velocity profile as the real flow, the fine scale turbulence merely contributing to eddy damping. However, as pointed out by Moore (1977), nonlinearities become significant for forcing levels as low as 0.1 % of the jet exit velocity. For the same reasons, the maximum value of the amplitude 'saturates' at higher forcing levels: whereas, in Crow & Champagne's experiment the total gain (i.e. the ratio of the maximum amplitude to the initial forcing amplitude) is 18 at 1 % forcing level, it is only 5 at 4 % forcing level.

Linear stability calculations such as those of Michalke (1971) and Crighton & Gaster (1976) have been very successful in predicting the value of the natural frequency associated with transitional as well as turbulent free shear flows. They have shown, in particular, that the experimentally measured dominant frequency is indeed the frequency of the most amplified wave as given by linear stability theory. In the case of the tanh y velocity profile and for spatially growing disturbances, the most unstable frequency suitably non-dimensionalized was found by Michalke (1965) to be 0.414. Ideally one would attempt to follow the growth of the linearly most unstable wave in the finite-amplitude régime, and thus obtain its maximum amplitude. However, attractive this approach may be, it is presently not amenable to theoretical treatment. In this study, we therefore restrict our attention to weakly amplified waves close to the neutral wavenumber or frequency. Stuart (1967) has shown that this type of analysis is valid only in the immediate vicinity of the neutral point. Hence, the present results cannot be freely extended to the most amplified wave and a direct comparison between theory and experiment it not justified unless the free shear layer is excited at a frequency sufficiently close to neutral stability.

From a theoretical point of view, the present investigation is closely related to the temporal nonlinear stability analysis of a parallel free shear layer carried out by Schade (1964). His approach relies on three basic assumptions:

(1) the amplitude of the fluctuations is small of order ϵ.

(2) the Reynolds number R scaled on the shear layer thickness is large.

(3) the waves are weakly amplified, in the sense that one considers wavenumbers in the vicinity of the neutral wavenumber. This restriction is essential if one is to be able to balance linear amplification by nonlinear effects.

If the real wavenumber K is within $O(\epsilon^2)$ of the neutral wavenumber K_n and such that

$$K = K_n - \Delta K \epsilon^2 \tag{1.1}$$

Schade's result can be expressed in terms of an evolution equation for the amplitude A of the fluctuations:

$$\frac{\mathrm{d}A}{\mathrm{d}T_2} = \frac{2}{\pi}(\Delta K - \tfrac{4}{3}|A|^2)\,A \tag{1.2}$$

where T_2 is a suitably defined slow time scale. Since the Landau constant multiplying the non-linear term is negative, this evolution implies the existence of a finite equilibrium amplitude for weakly amplified waves below the neutral wavenumber. Such a conclusion is in qualitative agreement with the experiments mentioned previously, and similar results have also been obtained by Benney & Maslowe (1975) and Huerre (1977).

It is important to draw attention to some of the restrictive assumptions that have been implicitly made in order to arrive at the evolution equation (1.2): in all these investigations a logarithmic singularity occurs at the origin and, following a common practice in linear stability theory, one chooses the branch of the logarithmic function which yields a $-\pi$ phase shift as one crosses the origin from below in the complex plane of the transverse coordinate y. Hence, one assumes that the critical layer is viscous of thickness $(KR)^{-\frac{1}{3}}$. However, viscous effects are entirely neglected outside the critical layer. Moreover, the mean flow change induced by Reynolds stresses is effectively taken to be zero so that the Landau constant in (1.2) only represents the exchange of energy between the fundamental and the harmonic fluctuations.

More recent studies provide considerable insight on the problems raised by this 'almost inviscid' approach. Benney & Bergeron (1969) showed how one could introduce nonlinear effects, instead of viscous effects, in the critical layer, in order to 'smooth out' the singularity arising at the critical point. Furthermore, by defining a Reynolds number in the critical layer $R_{\text{c.l.}} = R\epsilon^{\frac{3}{2}}$ based on its thickness $\epsilon^{\frac{1}{2}}$, Haberman (1972) was able to follow the increase of the phase shift from $-\pi$ to zero as $R_{\text{c.l.}}$ changes from zero to infinity. This study emphasized the role of $R_{\text{c.l.}}$ in providing a measure of the respective magnitude of nonlinear and viscous effects in the critical layer.

From the investigations of Benney & Maslowe (1975) and Huerre (1977) which were more specifically concerned with the nonlinear stability of the $\tanh y$ profile, it can also be concluded that the evolution equation (1.2) for the amplitude function is $R_{\text{c.l.}}$ dependent and becomes of 2nd order in time as $R_{\text{c.l.}}$ goes to infinity in the nonlinear critical layer régime. The Stuart–Watson approach has also been applied to the $\tanh y$ profile by Maslowe (1977), for finite shear layer Reynolds numbers, R, i.e. small $R_{\text{c.l.}}$. Maslowe calculated the part of the Landau constant pertaining to the harmonic numerically and showed that it decreased with increasing Reynolds number R. A rough estimate of the effect of Reynolds stresses indicated that mean flow distortion might counteract this trend.

A different approach has been proposed by Stuart (1967): instead of following the evolution of disturbances in time or in the downstream direction, Stuart calculated the inviscid finite-amplitude equilibrium state associated with the hyperbolic tangent profile. The resulting expansion was found to be regular at the origin, and this method, except for subtle differences discussed in the paper, led to a satisfactory agreement with Schade's result.

Recently, Stewartson (1978) and Brown & Stewartson (1978) carried out a detailed investigation of the dynamics of the critical layer as a weak Rossby wave is forced on a uniform shear for small and large values of $R_{c.l.}$. The evolution of the wave was followed for all time and some of the results which they obtained for a viscous critical layer will be discussed in § 5.

In this work, we limit ourselves to the question of the long time evolution of weakly amplified free modal disturbances and we solely consider the case of a viscous critical layer as in Schade's paper. Consequently, the shear layer Reynolds number R will be large but of finite order with respect to $1/\epsilon$ in order to ensure that $R_{c.l.}$ is small. As soon as this assumption is made, however, viscosity cannot be neglected outside the critical layer. This is indeed a necessary prerequisite if the method of matched asymptotic expansions is to be applied successfully to the flow in the critical layer. The distortion of the mean flow will also be taken into account and it will be shown that its contribution to the Landau constant does not lead to a finite amplitude equilibrium state.

Since R is not infinite, the tanh y velocity profile is not an exact solution of the basic flow equations. As discussed in the conclusion, fundamental difficulties arise when one attempts to relax the parallel flow assumption. In this investigation, we have chosen to introduce an artificial body force in order to counteract the effect of viscous diffusion on the basic flow. Such a procedure is not uncommon: Crow (1968) in his visco-elastic model of turbulence, used random body forces to maintain turbulence against viscous dissipation. It must be emphasized that a *body force was bound to be implicitly present* in all previous investigations concerning the linear and nonlinear stability of the tanh y profile at large or finite Reynolds numbers. In this study we merely introduce this body force explicitly. Thus, it should be made clear that comparison with earlier work is legitimate.

Under these assumptions, we seek to describe the amplitude evolution of weakly amplified waves as they grow in time or space. The basic equations are presented in § 1 and the outer expansion as well as the amplitude and mean flow distortion equations are partially derived in § 3, within the framework of the method of multiple scales. The goal of § 4 is to determine the characteristics of the inner critical layer so as to relate the outer flows above and below the origin. The problem has to be solved to $O(\epsilon^{\frac{10}{3}})$ and we have tried to shield most details from the reader. It was felt, however, that an outline of the calculations was necessary in order to derive the amplitude equations in a rigorous manner. Section 5 contains a discussion of the main results of the inner problem inasmuch as they affect the outer mean flow distortion and the velocity fluctuations. Finally, the amplitude equations and mean flow equation are derived in § 6 and their physical meaning is discussed in the particular case of temporally and spatially growing waves. We conclude the paper with some qualitative implications of this work to the question of the general nonlinear evolution of instability waves in free shear flows.

2. Basic equations and assumptions

The flow is assumed to be incompressible and two-dimensional, and governed by the two-dimensional vorticity equation

$$\frac{\partial}{\partial t}\nabla^2\Psi + J_0(\nabla^2\Psi, \Psi) = \frac{1}{R}\nabla^2\nabla^2\Psi + \frac{1}{R}\nabla^2 a \tag{2.1}$$

where $J_0(f, g)$ is the Jacobian operator

$$J_0(f,g) = \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial \xi}. \tag{2.2}$$

The total stream function Ψ and the independent variables ξ, y, and t have been non-dimensionalized in terms of the velocity U_0 of the basic flow at $y = \pm\infty$ and a typical length scale L of the same order of magnitude as the shear layer thickness. R is the Reynolds number $U_0\nu/L$ based on the above velocity and length scales and on the kinematic viscosity ν, and it is assumed to be large. The function $a(\xi, y)$ is the z component of a vector potential associated with a body force

$$\boldsymbol{f} = \frac{1}{R}\left(\frac{\partial a}{\partial y}\boldsymbol{e}_\xi - \frac{\partial a}{\partial \xi}\boldsymbol{e}_y\right),$$

to be determined later in the course of the analysis.

In a standard fashion the total stream function is decomposed into a basic stream function $\overline{\psi}(y)$ and a perturbation stream function ψ so that

$$\Psi(\xi, y, t) = \overline{\psi}(y) + \epsilon\psi(\xi, y, t), \tag{2.3}$$

where ϵ is a small parameter characterizing the magnitude of the fluctuations. In all the analysis, it is understood that $\overline{\psi}(y) = \ln(\cosh y)$ is the basic stream function associated with the velocity profile $\overline{U}(y) = \tanh y$.

In order to pursue the analysis, one needs to specify the order of magnitude of $1/R$ as compared to ϵ. Since the purpose of this investigation is to consider the particular situation where the critical layer is viscous, any scaling for $1/R$ must be such that the Reynolds number in the critical layer $R\epsilon^{\frac{3}{2}}$, based on its thickness $\epsilon^{\frac{1}{2}}$ and the magnitude of the perturbation velocity ϵ, is smaller than unity.

For convenience, it will be assumed that

$$1/R = \lambda\epsilon, \quad \lambda = O(1), \tag{2.4}$$

in which case the Reynolds number in the critical layer is of order $\epsilon^{\frac{1}{2}}$. This particular choice of scaling allows any viscous corrections in the outer flow to occur at the same order as nonlinear terms. A smaller value of R (with respect to $1/\epsilon$) would merely enhance viscous effects in the outer flow at the expense of nonlinear interactions. The above scaling has the advantage of 'maximizing' nonlinear interactions without destroying the viscous character of the critical layer.

The requirement that the critical layer be viscous has two main consequences:

First, the Reynolds number R cannot be made as large as possible so as to keep the basic flow parallel, and we have introduced a body force in order to enforce this condition. Secondly, in contrast to the implicit assumption of Schade (1964) and Huerre (1977), the effect of viscosity on the instability wave is non negligible outside the critical layer, and we will take it into account in the present study.

Before proceeding to the weakly nonlinear case, it is probably useful to recall the main results of linear stability theory for the hyperbolic tangent free shear layer, as carried out by Betchov & Szewczyk (1963), Michalke (1964) and Tatsumi *et al.* (1964). For temporally growing waves, the real part of the phase velocity is identically zero, and its imaginary part c_i is a function of the wavenumber K as shown in figure 1. As the Reynolds number decreases, the unstable range of wavenumbers becomes narrower, but there is no critical Reynolds number below which the shear layer is stable. For infinite Reynolds number, the neutral wavenumber separating the stable range from the unstable range is unity, and the associated eigenfunction is known explicitly and given by

$$\phi(y) = \operatorname{sech} y. \tag{2.5}$$

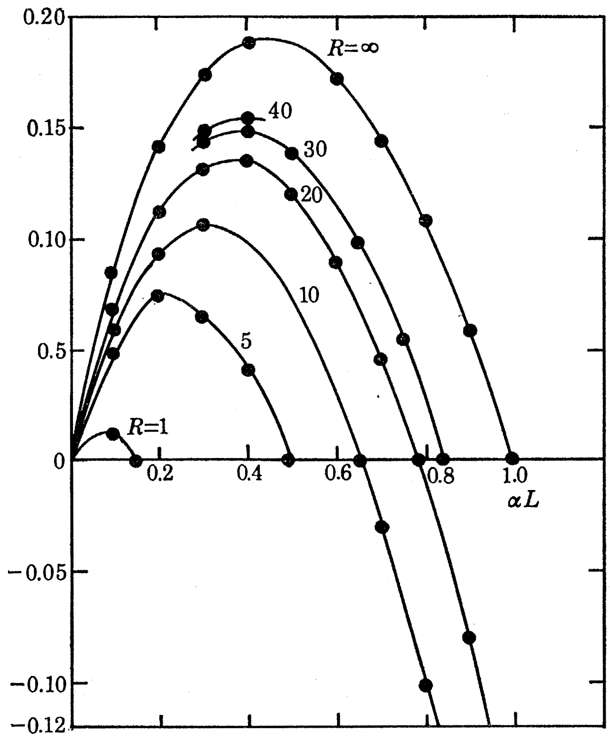

FIGURE 1. Amplification rate Kc_i against K for different values of the Reynolds number R (from Betchov & Szewczyk 1963; courtesy of *Physics of Fluids*).

Following Schade (1964) and Benney & Maslowe (1975), we take advantage of this feature and investigate the nonlinear behaviour of a wavetrain of wavenumber unity whose amplitude is slowly modulated in space and time. Equivalently, we seek to determine the nonlinear regime of weakly amplified modes whose wavenumber (for time growing waves) or frequency (for space growing waves) is close to the neutral wavenumber or neutral frequency. It is convenient to use the formalism of the method of multiple scales, as described in Nayfeh (1973), and introduce a set of slow time and space scales

$$T_1 = \epsilon t; \quad T_2 = \epsilon^2 t; \quad X_1 = \epsilon\xi; \quad X_2 = \epsilon^2\xi \tag{2.6}$$

so that partial derivatives in t and ξ become

$$\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon\frac{\partial}{\partial T_1} + \epsilon^2\frac{\partial}{\partial T_2}; \quad \frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial X_1} + \epsilon^2\frac{\partial}{\partial X_2}. \tag{2.7}$$

In contrast with previous weakly nonlinear studies, two sets of slow scales are needed in this instance. From a formal point of view, they are introduced in order to prevent the appearance of secular terms in the higher order approximations, namely the $O(\epsilon^2)$ and $O(\epsilon^3)$ solutions. The scales T_1 and X_1 are effectively viscous scales which will account for the slight decrease in amplification rate due to finite Reynolds number effects. The scales T_2 and X_2 are associated with nonlinear interactions in the same fashion as in other weakly nonlinear investigations.

The perturbation stream function ψ is then considered as a function of these slow scales and expanded in powers of ϵ

$$\psi(y, \xi, t, X_i, T_i) = \psi_1(y, \xi, t, X_i, T_i) + \epsilon\psi_2(y, \xi, t, X_i, T_i) + \epsilon^2\psi_3(y, \xi, t, X_i, T_i) + \dots. \tag{2.8}$$

Substitutions of (2.8) into the vorticity equation (2.1) leads to the equations determining ψ_1, ψ_2 and ψ_3. The next section considers the resulting outer flow outside the critical layer and derives the resulting amplitude and mean flow correction equations. It is convenient to introduce the following notations

$$J_i(f, g) = \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial X_i}, \quad i = 1, 2, \tag{2.9}$$

$$\mathscr{L}_0[\psi] = \left(\frac{\partial}{\partial t} + \overline{U}(y)\frac{\partial}{\partial \xi}\right)\nabla^2\psi - \overline{U}''(y)\frac{\partial \psi}{\partial \xi}, \tag{2.10}$$

$$\mathscr{L}_i[\psi] = \left(\frac{\partial}{\partial T_i} + \overline{U}(y)\frac{\partial}{\partial X_i}\right)\nabla^2\psi - \overline{U}''(y)\frac{\partial \psi}{\partial X_i}, \quad i = 1, 2. \tag{2.11}$$

3. Outer problems

First order problem. The governing equation for ψ_1 is

$$\mathscr{L}_0[\psi_1] = \lambda\overline{U}'''(y) + \lambda\nabla^2 a \tag{3.1}$$

and one is immediately faced with the divergence of the basic flow due to viscosity which was mentioned earlier. We shall choose the body force so that the right-hand side of (3.1) vanishes, and take

$$a = -\overline{U}'(y) \tag{3.2}$$

or

$$\boldsymbol{f} = -\lambda\epsilon\overline{U}''(y)\,\boldsymbol{e}_\xi. \tag{3.3}$$

For waves periodic in ξ, equation (3.1) then becomes the Rayleigh equation and we shall take as the solution to the first order problem the neutral wave

$$\psi_1 = \operatorname{sech} y \operatorname{Re} A(X_i, T_i)\,e^{i\xi}, \tag{3.4}$$

where $A(X_i, T_i)$ is an amplitude function to be determined in the second and third order problems.

Second order problem. The corresponding equation for ψ_2 is

$$\mathscr{L}_0[\psi_2] = -2\left(\frac{\partial}{\partial t} + \overline{U}(y)\frac{\partial}{\partial \xi}\right)\frac{\partial^2\psi_1}{\partial\xi\,\partial X_1} - \mathscr{L}_1[\psi_1] + \lambda\nabla^2\nabla^2\psi_1 + J_0(\psi_1, \nabla^2\psi_1) \tag{3.5}$$

and suggests a solution of the form

$$\psi_2 = \Phi_2^{(0)}(y, X_i, T_i) + \operatorname{Re}\Phi_2^{(1)}(y, X_i, T_i)\,e^{i\xi} + \operatorname{Re}\Phi_2^{(2)}(y, X_i, T_i)\,e^{2i\xi}. \tag{3.6}$$

The second term in (3.6) represents a modification to the fundamental and satisfies the following equation,

$$L^{(1)}[\Phi_2^{(1)}] = -2\mathrm{i}\,\mathrm{sech}\,y\frac{\partial A}{\partial X_1} - 2\mathrm{i}\,\mathrm{sech}^3 y\,\coth y\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) + 24\mathrm{i}\lambda\,\mathrm{sech}^3 y\,\tanh y A, \tag{3.7}$$

where
$$L^{(n)}[\phi] = \frac{\partial^2\phi}{\partial y^2} - \left(n^2 + \frac{\overline{U}''(y)}{\overline{U}(y)}\right)\phi, \quad n = 1, 2, \ldots. \tag{3.8}$$

In such cases it is customary to enforce an orthogonality condition of the form

$$\int_{-\infty}^{+\infty} Q_2^{(1)}(y, X_i, T_i)\,\phi_a(y)\,\mathrm{d}y = 0, \tag{3.9}$$

where $Q_2^{(1)}(y, X_i, T_i)$ is the forcing term and $\phi_a(y)$ is the eigenfunction (in this case $\mathrm{sech}\,y$) associated with the homogeneous equation

$$L^{(1)}[\phi_a] = 0. \tag{3.10}$$

The forcing term of equation (3.7) exhibits a singularity in $1/y$ so that the interpretation of the integral is ambiguous. In similar instances, Schade (1964) and Huerre (1977) used well known results of linear stability theory and integrated (3.9) below the singularity $y = 0$. Thus, they avoided any detailed consideration of the critical layer around the origin and assumed that, when the critical layer is viscous it is legitimate to choose for the logarithm arising in the integration of (3.9), the branch cut which goes from 0 to $+\mathrm{i}\infty$. This argument will not be used here and the second order outer problem will be solved on each side of the critical layer, the jump relations being determined in § 4 by matching across the origin. Hence, by a straightforward application of the method of variation of parameters, the solution of equation (3.7) is found to be

$$\begin{aligned}\Phi_2^{(1)}(y, X_i, T_i) = &-\mathrm{i}[(y\,\mathrm{sech}\,y + \sinh y)\ln|\tanh y| - \mathrm{sech}\,y\,\chi_2(\tanh y)][\partial A/\partial T_1 + 4\lambda A]\\ &-\mathrm{i}\tanh y\,\sinh y\,\partial A/\partial X_1 - 6\mathrm{i}\lambda\,\mathrm{sech}\,y\,\tanh y A + a_2^{(1)\pm}(X_i, T_i)\,\mathrm{sech}\,y\\ &+ b_2^{(1)\pm}(X_i, T_i)\,(y\,\mathrm{sech}\,y + \sinh y)\end{aligned} \tag{3.11}$$

where $a_2^{(1)\pm}(X_i, T_i)$ and $b_2^{(1)\pm}(X_i, T_i)$ are 2 unknown 'constants' multiplying the complementary solutions of (3.7), the + and − superscript corresponding to the outer region above and below $y = 0$ respectively. $\chi_2(x)$ is the inverse hyperbolic tangent integral defined by

$$\chi_2(x) = \int_0^x \frac{\mathrm{artanh}\,t}{t}\,\mathrm{d}t. \tag{3.12}$$

In order for $\Phi_2^{(1)}$ to vanish exponentially at $\pm\infty$, one must have

$$b_2^{(1)+}(X_i, T_i) = -b_2^{(1)-}(X_i, T_i) = \mathrm{i}\,\partial A/\partial X_1 \tag{3.13}$$

and the first amplitude equation describing the variations of A with X_1 and T_1 can be written as

$$2\mathrm{i}\,\partial A/\partial X_1 = b_2^{(1)+}(X_i, T_i) - b_2^{(1)-}(X_i, T_i). \tag{3.14}$$

The first harmonic in (3.6) which is generated by nonlinear interactions was already computed by Schade (1964) and is given by

$$\Phi_2^{(2)}(y, X_i, T_i) = -\tfrac{1}{4}\,\mathrm{sech}^4 y\,A^2. \tag{3.15}$$

Finally the mean flow correction $\Phi_2^{(0)}(y, X_i, T_i)$ is unknown and will be determined by the third order problem.

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Third order problem. The governing equation is

$$\mathscr{L}_0[\psi_3] = -2\left(\frac{\partial}{\partial t}+\bar{U}(y)\frac{\partial}{\partial \xi}\right)\frac{\partial^2\psi_2}{\partial\xi\,\partial X_1} - \mathscr{L}_1[\psi_2] + \lambda\nabla^2\nabla^2\psi_2 + J_0[\psi_1,\nabla^2\psi_2] + J_0[\psi_2,\nabla^2\psi_1]$$
$$-\mathscr{L}_2[\psi_1] - \left(\frac{\partial}{\partial t}+\bar{U}(y)\frac{\partial}{\partial \xi}\right)\left(\frac{\partial^2\psi_1}{\partial X_1^2}+2\frac{\partial^2\psi_1}{\partial\xi\,\partial X_2}\right)+4\lambda\frac{\partial^2\nabla^2\psi_1}{\partial\xi\,\partial X_1}$$
$$-2\left(\frac{\partial}{\partial T_1}+\bar{U}(y)\frac{\partial}{\partial X_1}\right)\frac{\partial^2\psi_1}{\partial\xi\,\partial X_1} + J_1[\psi_1,\nabla^2\psi_1] + 2J_0\left[\psi_1,\frac{\partial^2\psi_1}{\partial\xi\,\partial X_1}\right]. \quad (3.16)$$

The forcing term in this equation contains expressions which are independent of the ξ coordinate, and in order to prevent the appearance of secular terms in ψ_3, one must ensure that they are identically zero, which provides the equation for the mean flow correction

$$\left(\frac{\partial}{\partial T_1}+\bar{U}(y)\frac{\partial}{\partial X_1}\right)\frac{\partial^2\Phi_2^{(0)}}{\partial y^2}+\bar{U}''(y)\frac{\partial\Phi_2^{(0)}}{\partial X_1}-\lambda\frac{\partial^4\Phi_2^{(0)}}{\partial y^4}$$
$$=\tfrac{1}{2}\operatorname{sech}^4 y(3+\coth^2 y)\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)+\operatorname{sech}^4 y\tanh y\frac{\partial|A|^2}{\partial X_1}$$
$$-12\lambda\operatorname{sech}^4 y(5\tanh^2 y-1)\,|A|^2. \quad (3.17)$$

A first integral is immediately found in the form

$$\left(\frac{\partial}{\partial T_1}+\bar{U}(y)\frac{\partial}{\partial X_1}\right)\frac{\partial\Phi_2^{(0)}}{\partial y}-\bar{U}'(y)\frac{\partial\Phi_2^{(0)}}{\partial X_1}-\lambda\frac{\partial^3\Phi_2^{(0)}}{\partial y^3}$$
$$=-\tfrac{1}{2}\operatorname{sech}^4 y\coth y\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)-\tfrac{1}{4}\operatorname{sech}^4 y\frac{\partial|A|^2}{\partial X_1}$$
$$+12\lambda\operatorname{sech}^4 y\tanh y\,|A|^2-\alpha^{\pm}(X_i,T_i). \quad (3.18)$$

As explicitly shown in appendix A, this relation is precisely the equation of motion pertaining to the mean flow change, the first three terms on the right-hand side of (3.18) representing the action of Reynolds stresses. Furthermore, the constant of integration $\alpha^{\pm}(X_i,T_i)$ is then readily identified as the change in mean pressure gradient at $y=\pm\infty$ so that the mean flow equation can be cast in the final form

$$\left(\frac{\partial}{\partial T_1}+\bar{U}(y)\frac{\partial}{\partial X_1}\right)\frac{\partial\Phi_2^{(0)}}{\partial y}-\bar{U}'(y)\frac{\partial\Phi_2^{(0)}}{\partial X_1}-\lambda\frac{\partial^3\Phi_2^{(0)}}{\partial y^3}$$
$$=-\tfrac{1}{2}\operatorname{sech}^4 y\coth y\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)-\tfrac{1}{4}\operatorname{sech}^4 y\frac{\partial|A|^2}{\partial X_1}$$
$$+12\lambda\operatorname{sech}^4 y\tanh y\,|A|^2-\frac{\partial\bar{P}_{2\infty}^{(0)\pm}}{\partial X_1}. \quad (3.19)$$

The form of the ξ dependent terms on the right-hand side of (3.16) suggests a third order solution given by

$$\psi_3=\Phi_3^{(0)}(y,X_i,T_i)+\operatorname{Re}\Phi_3^{(1)}(y,X_i,T_i)\,e^{i\xi}+\operatorname{Re}\Phi_3^{(2)}(y,X_i,T_i)\,e^{2i\xi}+\operatorname{Re}\Phi_3^{(3)}(y,X_i,T_i)\,e^{3i\xi}. \quad (3.20)$$

The correction to the fundamental $\Phi_3^{(1)}$ is then found to satisfy the following equation

$$L^{(1)}[\Phi_3^{(1)}]=Q_3^{(1)}(y,X_i,T_i). \quad (3.21)$$

Calculations become very extensive at this point of the analysis and the reader is referred to appendix B for a detailed expression of the forcing term $Q_3^{(1)}(y,X_i,T_i)$. It is important to note, however, that $Q_3^{(1)}$ has a $1/y^4$ singularity at the origin. Invoking an orthogonality condition of the type (3.9) therefore becomes even more delicate than in the second order problem since the integral does not exist as a Cauchy principal value. Nevertheless, a modified 'solvability' condition can be derived from the application of the boundary conditions as suggested in similar circumstances by Benney & Maslowe (1975) and by Redekopp (1977). The procedure is now briefly explained in this particular case:

If $\phi_a(y) = \operatorname{sech} y$ and $\phi_b(y) = y \operatorname{sech} y + \sinh y$ are the two complementary functions associated with $L^{(1)}$, the general solution of equation (3.21) formally becomes

$$\Phi_3^{(1)}(y, X_i, T_i) = \frac{1}{W}\int^y Q_3^{(1)}(\eta, X_i, T_i)\,[\phi_a(\eta)\,\phi_b(y) - \phi_b(\eta)\,\phi_a(y)]\,\mathrm{d}\eta + a_3^{(1)\pm}(y, X_i, T_i)\,\phi_a(y) + b_3^{(1)\pm}(y, X_i, T_i)\,\phi_b(y), \tag{3.22}$$

where W is the Wronskian of $\phi_a(y)$ and $\phi_b(y)$, and $a_3^{(1)\pm}$, $b_3^{(1)\pm}$ are unknown 'constants'. It can be checked that $Q_3^{(1)}(y, X_i, T_i) \sim \mathrm{e}^{\mp y}$ as $y \to \pm\infty$, so that in order for $\Phi_3^{(1)}(y, X_i, T_i)$ to vanish exponentially at infinity we must have

$$I(+\infty) - I(-\infty) = W(b_3^{(1)-} - b_3^{(1)+}), \tag{3.23}$$

where $I(y)$ is the indefinite integral

$$I(y) = \int^y Q_3^{(1)}(\eta, X_i, T_i)\,\phi_a(\eta)\,\mathrm{d}\eta. \tag{3.24}$$

Hence, with the use of this modified solvability condition, the explicit determination of $\Phi_3^{(1)}$ can be entirely avoided. Substitution of $Q_3^{(1)}(y, X_i, T_i)$ given in appendix B leads after some calculations to the following amplitude equation

$$\begin{aligned}2\mathrm{i}\frac{\partial A}{\partial X_2} - \frac{\partial^2 A}{\partial X_1^2} &+ (3 - 4\chi_3(1))\frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) + 24\lambda\frac{\partial A}{\partial T_1} + 4\lambda(\tfrac{11}{3} - 4\chi_3(1))\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) + 192\lambda^2 A \\ &- \tfrac{4}{3}|A|^2 A - \tfrac{1}{2}[B(+\infty, X_i, T_i) - B(-\infty, X_i, T_i)]\,A - 5\mathrm{i}\lambda(a_2^{(1)+} - a_2^{(1)-}) \\ &+ 4\mathrm{i}\lambda(1 + \chi_3(1))\,(b_2^{(1)+} + b_2^{(1)-}) + \mathrm{i}\frac{\partial}{\partial X_1}(a_2^{(1)+} + a_2^{(1)-}) - \tfrac{1}{2}\mathrm{i}\frac{\partial}{\partial T_1}(a_2^{(1)+} - a_2^{(1)-}) \\ &+ \tfrac{1}{2}\mathrm{i}(1 + 2\chi_2(1))\frac{\partial}{\partial T_1}(b_2^{(1)+} + b_2^{(1)-}) = b_3^{(1)+} - b_3^{(1)-},\end{aligned} \tag{3.25}$$

where the function $\chi_3(x)$ is the inverse hyperbolic tangent integral of third order defined by

$$\chi_3(x) = \int_0^x \frac{\chi_2(t)}{t}\,\mathrm{d}t \tag{3.26}$$

and the function $B(y, X_i, T_i)$ is the indefinite integral

$$B(y, X_i, T_i) = \int^y \operatorname{sech}^2 t \coth t\left(\frac{\partial^3 \Phi_2^{(0)}}{\partial t^3} + 2\operatorname{sech}^2 t\frac{\partial \Phi_2^{(0)}}{\partial t}\right)\mathrm{d}t. \tag{3.27}$$

In the second order problem, the amplitude equation (3.14) was derived from the full solution for $\Phi_2^{(1)}$. The reader may immediately verify that the same result could have been obtained by applying condition (3.24) to the forcing term $Q_2^{(1)}(y, X_i, T_i)$.

Finally, the correction to the first harmonic $\Phi_3^{(2)}$ obeys the equation

$$L^{(2)}[\Phi_3^{(2)}] = Q_3^{(2)}(y, X_i, T_i), \tag{3.28}$$

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where $Q_3^{(2)}(y, X_i, T_i)$ is written in detail in appendix B, and $L^{(2)}$ is the operator defined in (3.8) with $n = 2$. The second harmonic satisfies the equation

$$L^{(3)}[\Phi_3^{(3)}] = -\tfrac{5}{6}\operatorname{sech}^5 y\, A^3 \tag{3.29}$$

and is perfectly regular at this order.

The outer analysis has so far enabled us to characterise the variations of the function $A(X_i, T_i)$ by two amplitude equations (3.14) and (3.25) and the distribution of $\Phi_2^{(0)}(y, X_i, T_i)$ by the equation of motion (3.19). However the study is far from complete because one does not know the jump conditions which in particular determine $b_2^{(1)+} - b_2^{(1)-}$ and $b_3^{(1)+} - b_3^{(1)-}$. These are expected to result from the inner analysis of the problem around $y = 0$, and from the subsequent matching conditions with the outer flows on both sides of the critical layer. A first sign of the irregularity of the outer expansion around $y = 0$, is given in the second order problem by the presence of the $1/y$ singularity in $Q_2^{(1)}$ (see equation (3.7)) and the associated logarithmic singularity in $\Phi_2^{(1)}$ (see equation (3.11)). In the third order problem, the forcing term $Q_3^{(1)}$ is singular in $1/y^4$, and by comparison with $Q_2^{(1)}$, one can estimate the thickness of the critical layer to be $O(\epsilon^{\frac{1}{3}})$. This is also the scaling of the transverse coordinate y which will yield a viscous diffusion term and a mean flow correction term of equal order of magnitude in the vorticity equation.

Before proceeding to the study of the critical layer, it is important to define the detailed asymptotic behaviour around the origin of each term in the outer expansion so as to be able to apply the matching principle at each order in the inner problem. The procedure is rather tedious: a general form of the expansion is assumed and the coefficients are determined by identification after substitution into the governing equations given in this section. For instance, the mean flow correction $\Phi_2^{(0)}(y, X_i, T_i)$ admits the following expansion

$$\Phi_2^{(0)}(y, X_i, T_i) \sim a_2^{(0)\pm} + b_2^{(0)\pm} y + \left[c_2^{(0)\pm} + \frac{1}{4\lambda}\left(\frac{\partial |A|^2}{\partial T_1} + 8\lambda |A|^2\right)\ln|y|\right] y^2$$
$$+ \frac{1}{6\lambda}\left[\frac{\partial \bar{P}_{2\infty}^{(0)\pm}}{\partial X_1} + \frac{1}{4}\frac{\partial |A|^2}{\partial X_1} + \frac{\partial b_2^{(0)\pm}}{\partial T_1} - \frac{\partial a_2^{(0)\pm}}{\partial X_1}\right] y^3 + \dots \quad \text{as} \quad y \to 0^{\pm}, \tag{3.30}$$

where $a_2^{(0)\pm}$, $b_2^{(0)\pm}$ and $c_2^{(0)\pm}$ are multiplicative constants associated with the complementary functions of the mean flow equation (3.19). Similarily, the expansion for the distortion of the first harmonic $\Phi_3^{(2)}$ reads

$$\Phi_3^{(2)}(y, X_i, T_i) \simeq \tfrac{1}{8}\mathrm{i}\left(\frac{\partial A^2}{\partial T_1} + 8\lambda A^2\right)\frac{1}{y} + a_3^{(2)\pm} + \left[3b_3^{(2)\pm} + \mathrm{i}\left(\frac{\partial A^2}{\partial T_1} + 14\lambda A^2\right)\ln|y|\right] y + \dots \tag{3.31}$$

where $a_3^{(2)\pm}$ and $b_3^{(2)\pm}$ are associated with the complementary functions $\cosh^2 y$ and $\sinh 2y + \tanh y$ of the operator $L^{(2)}$ in equation (3.28). Jump conditions pertaining to these quantities will also be derived in the course of the inner study. All the other expansions do not involve the introduction of any new parameters and the results have been gathered in a separate appendix C. One may note that the outer expansion is only weakly singular so that the inner problem will have to be pursued to relatively high order before the required jump conditions on $b_3^{(1)\pm}$ can be established. The origin of this difficulty can be traced to the form of the 1st order solution. In the terminology of Redekopp (1977), we are considering the nonlinear behaviour of a regular neutral mode, as opposed to a singular neutral mode, and singularities only arise at higher order. The reader who is not interested in the details of the inner problem may without difficulty avoid the next section and directly proceed to § 5.

4. The inner critical layer

The length scale of the inner region is $O(\epsilon^{\frac{1}{3}})$ and the transverse coordinate y is rescaled according to the transformation

$$y = \epsilon^{\frac{1}{3}} Y. \tag{4.1}$$

It is also convenient to define the following inner operators

$$J_{*0}[f, g] = \frac{\partial f}{\partial \xi}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial \xi}, \tag{4.2}$$

$$J_{*i}[f, g] = \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X_i}, \quad \mathrm{i} = 1, 2, \tag{4.3}$$

$$\mathscr{L}_{*0}[\psi] = \lambda \frac{\partial^4 \psi}{\partial Y^4} - Y \frac{\partial^3 \psi}{\partial \xi \, \partial Y^2}, \tag{4.4}$$

whereby the vorticity equation (2.1) expressed in terms of $\Psi^*(Y, \xi, X_i, T_i)$ becomes

$$\begin{aligned}
\lambda \frac{\partial^4 \Psi^*}{\partial Y^4} - J_{*0}\left[\frac{\partial^2 \Psi^*}{\partial Y^2}, \Psi^*\right] = {} & \left\{\lambda \bar{U}'''(\epsilon^{\frac{1}{3}} Y) + \frac{\partial}{\partial T_1}\frac{\partial^2 \Psi^*}{\partial Y^2} + J_{*0}\left[\frac{\partial^2 \Psi^*}{\partial \xi^2}, \Psi^*\right] - 2\lambda \frac{\partial^4 \Psi^*}{\partial \xi^2 \partial Y^2}\right\}\epsilon^{\frac{2}{3}} \\
& + J_{*1}\left[\frac{\partial^2 \Psi^*}{\partial Y^2}, \Psi^*\right]\epsilon + \left(\frac{\partial^3 \Psi^*}{\partial \xi^2 \partial T_1} - \lambda \frac{\partial^4 \Psi^*}{\partial \xi^4}\right)\epsilon^{\frac{4}{3}} + \left\{\frac{\partial}{\partial T_2}\frac{\partial^2 \Psi^*}{\partial Y^2}\right. \\
& \left. + 2 J_{*0}\left[\frac{\partial^2 \Psi^*}{\partial \xi \, \partial X_1}, \Psi^*\right] + J_{*1}\left[\frac{\partial^2 \Psi^*}{\partial \xi^2}, \Psi^*\right] - 4\lambda \frac{\partial^4 \Psi^*}{\partial \xi \, \partial X_1 \partial Y^2}\right\}\epsilon^{\frac{5}{3}} \\
& + J_{*2}\left[\frac{\partial^2 \Psi^*}{\partial Y^2}, \Psi^*\right]\epsilon^2 + \left\{2\frac{\partial^3 \Psi^*}{\partial \xi \, \partial T_1 \partial X_1} + \frac{\partial^3 \Psi^*}{\partial \xi^2 \partial T_2} - 4\lambda \frac{\partial^4 \Psi^*}{\partial \xi^3 \partial X_1}\right\}\epsilon^{\frac{7}{3}} + O(\epsilon^{\frac{8}{3}}).
\end{aligned} \tag{4.5}$$

Furthermore, the stream function is expanded in powers of $\epsilon^{\frac{1}{3}}$ to read

$$\Psi^*(Y, \xi, X_i, T_i) = \tfrac{1}{2} Y^2 \epsilon^{\frac{2}{3}} + \psi_1^* \epsilon + \psi_{\frac{4}{3}}^* \epsilon^{\frac{4}{3}} + \psi_{\frac{5}{3}}^* \epsilon^{\frac{5}{3}} + \psi_2^* \epsilon^2 + \psi_{\frac{7}{3}}^* \epsilon^{\frac{7}{3}} + \psi_{\frac{8}{3}}^* \epsilon^{\frac{8}{3}} + \psi_3^* \epsilon^3 + \psi_{\frac{10}{3}}^* \epsilon^{\frac{10}{3}} + \dots. \tag{4.6}$$

Terms of the form $\epsilon^p (\ln \epsilon)^q$ will be formally included in ψ_p^* and, whenever they arise, we will use block-matching, thereby satisfying the criteria for a sound matching principle described by Lesser & Crighton (1975).

Substitution of the above expansion into the vorticity equation (4.5) leads to a sequence of inner problems which gradually become more and more tedious to solve. At each order the inner solution is matched with the outer solution by comparing the outer expansion of the inner expansion with the inner expansion of the outer expansion for $y > 0$ and $y < 0$ which are tabulated in appendix C. No special difficulties are encountered in the first few steps which are outlined below:

$O(\epsilon)$ problem

$$\mathscr{L}_{*0}[\psi_1^*] = 0. \tag{4.7}$$

The first order stream function is assumed to be of the form

$$\psi_1^* = \operatorname{Re} \Phi_1^{*(1)}(Y, X_i, T_i)\, \mathrm{e}^{\mathrm{i}\xi} \tag{4.8}$$

and, consequently, $\Phi_1^{*(1)}$ satisfies the ordinary differential equation

$$L^{*(1)}[\Phi_1^{*(1)}] = 0, \tag{4.9}$$

where the following notation has been introduced

$$L^{*(n)}[\phi] \equiv \lambda \frac{\partial^4 \phi}{\partial Y^4} - \mathrm{i} n Y \frac{\partial^2 \phi}{\partial Y^2}; \quad n = 1, 2, \ldots. \tag{4.10}$$

This equation which frequently arises in viscous critical layer theory admits four independent solutions

$$1; \quad Y; \quad \int_0^{\mathrm{i}\lambda^{-\frac{1}{3}} Y} \int_0^u \sqrt{t}\, H^{(1)}_{\frac{1}{3}}(\tfrac{2}{3} t^{\frac{3}{2}})\, \mathrm{d}t\, \mathrm{d}u; \quad \int_0^{\mathrm{i}\lambda^{-\frac{1}{3}} Y} \int_0^u \sqrt{t}\, H^{(2)}_{\frac{1}{3}}(\tfrac{2}{3} t^{\frac{3}{2}})\, \mathrm{d}t\, \mathrm{d}u; \tag{4.11}$$

The third and fourth solutions, however, cannot possibly be part of the inner solution since they increase exponentially at $+\infty$ and $-\infty$ respectively. On the basis of these remarks, the first order solution is found to be

$$\psi_1^* = \mathrm{Re}\, A(X_i, T_i)\, \mathrm{e}^{\mathrm{i}\xi}. \tag{4.12}$$

The governing equations and final solutions of the next three problems readily follow:

$O(\epsilon^{\frac{4}{3}})$ problem

$$\mathscr{L}_{*0}[\psi^*_{\frac{4}{3}}] = -2\lambda, \tag{4.13}$$

$$\psi^*_{\frac{4}{3}} = -\tfrac{1}{12} Y^4. \tag{4.14}$$

$O(\epsilon^{\frac{5}{3}})$ problem

$$\mathscr{L}_{*0}[\psi^*_{\frac{5}{3}}] = Y \mathrm{Re}\, \mathrm{i} A\, \mathrm{e}^{\mathrm{i}\xi}, \tag{4.15}$$

$$\psi^*_{\frac{5}{3}} = -\tfrac{1}{2} Y^2 \mathrm{Re}\, A(X_i, T_i)\, \mathrm{e}^{\mathrm{i}\xi}. \tag{4.16}$$

$O(\epsilon^2)$ problem

$$\mathscr{L}_{*0}[\psi^*_2] = 8\lambda Y^2, \tag{4.17}$$

$$\psi^*_2 = \tfrac{1}{45} Y^6 + a_2^{(0)} + \mathrm{Re}\, a_2^{(1)}\, \mathrm{e}^{\mathrm{i}\xi} - \tfrac{1}{4} \mathrm{Re}\, A^2\, \mathrm{e}^{2\mathrm{i}\xi}, \tag{4.18}$$

where $a_2^{(0)}$ and $a_2^{(1)}$ are the constants defined in the outer problem. Application of the matching principle at this order immediately shows that they have the same value above and below the origin so that

$$a_2^{(0)+} = a_2^{(0)-} = a_2^{(0)}, \tag{4.19}$$

$$a_2^{(1)+} = a_2^{(1)-} = a_2^{(1)}. \tag{4.20}$$

The next four steps of the inner study involve a lot of straightforward algebra. However, our purpose here is not so much to analyse the characteristics of the critical layer for themselves, as it is to derive the jump conditions pertaining to the outer problem. In certain cases we will therefore be satisfied with the determination of the asymptotic behaviour of the solutions for large Y, which is all that is needed in order to apply the matching principle.

$O(\epsilon^{\frac{7}{3}})$ problem

$$\mathscr{L}_{*0}[\psi^*_{\frac{7}{3}}] = -\mathrm{Re}\left(2\frac{\partial A}{\partial T_1} + 3\lambda A + \tfrac{5}{2}\mathrm{i} A Y^3\right) \mathrm{e}^{\mathrm{i}\xi}. \tag{4.21}$$

Matching of the mean flow terms suggests a solution of the form

$$\psi^*_{\frac{7}{3}} = b_2^{(0)} Y + \mathrm{Re}\, \Phi^*_{\frac{7}{3}}(Y, X_i, T_i)\, \mathrm{e}^{\mathrm{i}\xi}, \tag{4.22}$$

where

$$b_2^{(0)+} = b_2^{(0)-} = b_2^{(0)} \tag{4.23}$$

and $\Phi^{*(1)}_{\frac{7}{3}}$ satisfies the differential equation

$$L^{*(1)}[\Phi^{*(1)}_{\frac{7}{3}}] = -\left(2\frac{\partial A}{\partial T_1} + 3\lambda A + \tfrac{5}{2}\mathrm{i} A Y^3\right). \tag{4.24}$$

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By analogy with the work of Graebel (1966), we introduce the function

$$w(z) = \int_0^z \int_0^u (G_i(t) + \mathrm{i}A_i(t))\,\mathrm{d}t\,\mathrm{d}u, \tag{4.25}$$

where Gi(z) is defined by Luke (1962) to be a particular solution of

$$(d^2/dz^2 - z)\,G_i(z) = -\pi^{-1}. \tag{4.26}$$

A suitable solution of (4.24) which does not increase exponentially at $Y = \pm\infty$ can then be written as follows

$$\Phi^{*(1)}_{\frac{7}{3}} = \tfrac{5}{24}AY^4 - \pi\lambda^{\frac{1}{3}}(1+\mathrm{i}\sqrt{3})\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) w\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}} Y\right] + m^{(1)}_{\frac{7}{3}} + n^{(1)}_{\frac{7}{3}}\,Y, \tag{4.27}$$

where $m^{(1)}_{\frac{7}{3}}$ and $n^{(1)}_{\frac{7}{3}}$ are unknown constants to be determined by matching. By making use of Luke's asymptotic formula for the integrals of Gi(z) and Ai(z), one can show that

$$w(z) \sim \frac{z}{\pi}\left[\ln z + \tfrac{1}{3}(\ln 3 + 2\gamma - 3 + \mathrm{i}\pi) + \frac{\pi 3^{-\frac{5}{6}}}{z\Gamma(\frac{1}{3})}(1-\mathrm{i}\sqrt{3}) + \frac{1}{3}\frac{1}{z^3} + \dots\right] \quad \text{as} \quad |z| \to \infty - \pi < \arg z < \tfrac{1}{3}\pi. \tag{4.28}$$

If we then only consider that part of the stream function which is associated with the fundamental, the outer expansion of the inner expansion carried out to order $\epsilon^{\frac{7}{3}}$ included, and rewritten in terms of the inner variable Y reads

$O(\epsilon^{\frac{7}{3}})$ outer $[O(\epsilon^{\frac{7}{3}})$ inner fundamental$]$

$$= A\epsilon - \tfrac{1}{2}AY^2\epsilon^{\frac{5}{3}} + a^{(1)}_2\epsilon^2 + \left\{\frac{5AY^4}{24} - 2\mathrm{i}\left[\frac{\partial A}{\partial T_1} + 4\lambda A\right][\tfrac{1}{3}(\ln 3 + 2\gamma + \tfrac{3}{2}\mathrm{i}\pi - \ln\lambda) + \ln Y]\,Y + n^{(1)}_{\frac{7}{3}}Y + m^{(1)}_{\frac{7}{3}} - 2\mathrm{i}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right)\frac{\pi 3^{-\frac{5}{6}}(1-\mathrm{i}\sqrt{3})}{\Gamma(\frac{1}{3})}\left(\frac{\lambda}{\mathrm{i}}\right)^{\frac{1}{3}}\right\}\epsilon^{\frac{7}{3}}. \tag{4.29}$$

Comparison with formula (C 11) leads to the jump condition

$$b^{(1)+}_2 - b^{(1)-}_2 = \pi(\partial A/\partial T_1 + 4\lambda A) \tag{4.30}$$

and enables us to determine the value of $m^{(1)}_{\frac{7}{3}}$ and $n^{(1)}_{\frac{7}{3}}$. The final result is

$$\Phi^{*(1)}_{\frac{7}{3}} = -\tfrac{2}{3}\mathrm{i}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) Y\ln\epsilon + \frac{5AY^4}{24} - 6\mathrm{i}\lambda AY - \pi\lambda^{\frac{1}{3}}(1+\mathrm{i}\sqrt{3})\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) \times\left\{w\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}} Y\right] - \frac{3^{-\frac{5}{6}}(1-\mathrm{i}\sqrt{3})}{\Gamma(\frac{1}{3})} - \frac{1}{6\pi\lambda^{\frac{1}{3}}}(\sqrt{3}+\mathrm{i})\,(\ln 3 + 2\gamma - \tfrac{3}{2} - \ln\lambda)\,Y\right\}. \tag{4.31}$$

This completes the problem of $O(\epsilon^{\frac{7}{3}})$.

$O(\epsilon^{\frac{8}{3}})$ *problem.* The general solution is taken to be of the form

$$\psi^*_{\frac{8}{3}} = \mathrm{Re}\,\Phi^{*(0)}_{\frac{8}{3}}(Y, X_i, T_i) + \mathrm{Re}\,\Phi^{*(1)}_{\frac{8}{3}}(Y, X_i, T_i)\,\mathrm{e}^{\mathrm{i}\xi} + \mathrm{Re}\,\Phi^{*(2)}_{\frac{8}{3}}(Y, X_i, T_i)\,\mathrm{e}^{2\mathrm{i}\xi}, \tag{4.32}$$

where the mean flow change of $O(\epsilon^{\frac{8}{3}})$ satisfies the following equation

$$\lambda\frac{\partial^4\Phi^{*(0)}_{\frac{8}{3}}}{\partial Y^4} = -\tfrac{34}{3}\lambda Y^4 + \frac{\pi}{2\lambda^{\frac{2}{3}}}(1+\mathrm{i}\sqrt{3})\,\bar{A}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) w'''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}} Y\right] \tag{4.33}$$

and has to match with the expression (C 6) at $Y=\pm\infty$. By using the same method as in the problem of $O(\epsilon^{\frac{7}{3}})$, one obtains the solution

$$\Phi^{*(0)}_{\frac{8}{3}} = -\frac{17Y^8}{2520}+\tfrac{1}{2}\pi\lambda^{-\frac{4}{3}}(1-\mathrm{i}\sqrt{3})\,\tilde{A}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)\int_0^{(\mathrm{i}/\lambda)^{\frac{1}{3}}Y} w(t)\,\mathrm{d}t-\tfrac{1}{2}\pi\lambda^{-\frac{4}{3}}\frac{1}{3^{\frac{7}{6}}\Gamma(\frac{2}{3})}\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda(A)^2\right)$$
$$-\frac{2\lambda^{-\frac{2}{3}}}{3^{\frac{5}{6}}\Gamma(\frac{1}{3})}\frac{\partial|A|^2}{\partial X_1}Y+\left[\frac{1}{12\lambda}\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)(\ln\epsilon-\ln 3-2\gamma+\ln\lambda+\tfrac{9}{2})+\tfrac{1}{2}(c_2^{(0)+}+c_2^{(0)-})\right]Y^2. \tag{4.34}$$

Matching with the outer flow also yields the jump condition

$$c_2^{(\;)+}-c_2^{(0)-} = -\frac{1}{2\lambda}\frac{\partial|A|^2}{\partial X_1}. \tag{4.35}$$

The contribution $\Phi^{*(1)}_{\frac{8}{3}}$ to the fundamental is governed by the equation,

$$L^{*(1)}[\Phi^{*(1)}_{\frac{8}{3}}] = (\mathrm{i}a_2^{(1)}-2\partial A/\partial X_1)\,Y, \tag{4.36}$$

subject to the matching condition (C 12). It is given by

$$\Phi^{*(1)}_{\frac{8}{3}} = -(a_2^{(1)}+\mathrm{i}\partial A/\partial X_1)\,Y^2. \tag{4.37}$$

Finally, the first harmonic is determined by

$$L^{*(2)}[\Phi^{*(2)}_{\frac{8}{3}}] = -\mathrm{i}A^2Y-\frac{\pi}{4\lambda^{\frac{2}{3}}}(1+\mathrm{i}\sqrt{3})\left(\frac{\partial A^2}{\partial T_1}+8\lambda A^2\right)w'''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right]. \tag{4.38}$$

subject to the matching condition (C 17) and is found to be of the following form

$$\Phi^{*(2)}_{\frac{8}{3}} = \frac{\mathrm{A}^2Y^4}{4}-\frac{\pi(1-\mathrm{i}\sqrt{3})}{4\lambda^{\frac{1}{3}}}\left(\frac{\partial A^2}{\partial T_1}+8\lambda A^2\right)w_1\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right], \tag{4.39}$$

where $w_1(z)$ is the particular solution of

$$\frac{\mathrm{d}^4w_1}{\mathrm{d}z^4}-2z\frac{\mathrm{d}^2w_1}{\mathrm{d}z^2} = G_i'(z)+\mathrm{i}A_i'(z) \tag{4.40}$$

which is such that

$$w_1(z)\sim\frac{\pi^{-1}}{4z}\left(1+\frac{7}{5}\frac{1}{z^3}+\dots\right)\quad\text{as}\quad |z|\to\infty\quad -\pi<\arg z<\tfrac{1}{3}\pi. \tag{4.41}$$

$O(\epsilon^3)$ *problem.* It is convenient to write the solution in the form

$$\psi_3^* = \mathrm{Re}\,\Phi_3^{*(0)}(Y,X_i,T_i)+\mathrm{Re}\,\Phi_3^{*(1)}(Y,X_i,T_i)\,\mathrm{e}^{\mathrm{i}\xi}$$
$$+\mathrm{Re}\,\Phi_3^{*(2)}(Y,X_i,T_i)\,\mathrm{e}^{2\mathrm{i}\xi}+\mathrm{Re}\,\Phi_3^{*(3)}(Y,X_i,T_i)\,\mathrm{e}^{3\mathrm{i}\xi} \tag{4.42}$$

where the mean flow terms obey the simple equation

$$\partial^4\Phi_3^{*(0)}/\partial Y^4 = 0 \tag{4.43}$$

together with the matching condition (C 7). The solution is

$$\Phi_3^{*(0)} = a_3^{(0)}+\frac{1}{6\lambda}\left(\frac{\partial\bar{P}^{(0)}_{2\infty}}{\partial X_1}+\frac{1}{4}\frac{\partial|A|^2}{\partial X_1}+\frac{\partial b_2^{(0)}}{\partial T_1}-\frac{\partial a_2^{(0)}}{\partial X_1}\right)Y^3 \tag{4.44}$$

and the following identities are derived by matching with the outer flow

$$\frac{\partial\bar{P}^{(0)+}_{2\infty}}{\partial X_1} = \frac{\partial\bar{P}^{(0)-}_{2\infty}}{\partial X_1} = \frac{\partial\bar{P}^{(0)}_{2\infty}}{\partial X_1}, \tag{4.45}$$

$$a_3^{(0)+} = a_3^{(0)-} = a_3^{(0)}. \tag{4.46}$$

The fundamental $\Phi_3^{*(1)}$ is determined by the equation

$$L^{*(1)}[\Phi_3^{*(1)}] = \tfrac{61}{24}\mathrm{i}AY^5 - \frac{\lambda AY^2}{2} + \frac{2}{3}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)Y^2\ln\epsilon + 3Y^2\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)$$
$$-\frac{\pi\lambda^{-\frac{1}{3}}}{3}(\sqrt{3}+\mathrm{i})\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)Y^3 w''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] + \pi\lambda^{\frac{1}{3}}(\sqrt{3}-\mathrm{i})\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)Y$$
$$\times\left\{w\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] - \frac{3^{-\frac{5}{6}}(1-\mathrm{i}\sqrt{3})}{\Gamma(\frac{1}{3})} - \frac{1}{6\pi\lambda^{\frac{1}{3}}}(\sqrt{3}+\mathrm{i})\,(\ln 3+2\gamma-\tfrac{3}{2}-\ln\lambda)\,Y\right\}$$
$$+2\pi\lambda^{\frac{2}{3}}(1-\mathrm{i}\sqrt{3})\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)w''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] - \frac{\pi\lambda^{-\frac{4}{3}}}{4}\left\{(1+\mathrm{i}\sqrt{3})\,A^2\left(\frac{\partial\tilde{A}}{\partial T_1}+4\lambda\tilde{A}\right)\right.$$
$$\left.\times\,\tilde{w}''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] - (1-\mathrm{i}\sqrt{3})\,|A|^2\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)w''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right]\right\} + \pi\lambda^{-\frac{1}{3}}(1-\mathrm{i}\sqrt{3})$$
$$\times\frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)w''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] + \frac{\pi\lambda^{-\frac{4}{3}}}{8}(1-\mathrm{i}\sqrt{3})\,\tilde{A}\left[\frac{\partial A^2}{\partial T_1}+8\lambda A^2\right]w_1''''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] \quad (4.47)$$

subject to the matching condition (C 13). It is very tedious but straightforward to show that $\Phi_3^{*(1)}$ is the particular solution of (4.47) which, as $|Y|\to\infty$, admits the expansion

$$\Phi_3^{*(1)} \sim -\tfrac{61}{720}AY^6 + \left[5\mathrm{i}\lambda A - \frac{\mathrm{i}}{18}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right) + \frac{\mathrm{i}}{3}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)\left(\tfrac{1}{3}\ln\epsilon+\frac{\mathrm{i}\pi}{2}+\ln Y\right)\right]Y^3$$
$$-\left[\frac{16\lambda}{3}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right) + 2\frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right) + \frac{1}{2\lambda}A\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)\right]\left[\tfrac{1}{3}\ln\epsilon+\frac{\mathrm{i}\pi}{2}+\ln Y\right]$$
$$+\frac{a_3^{(1)+}+a_3^{(1)-}}{2} + O\left(\frac{1}{Y^3}\right) \quad \text{when} \quad -\frac{7\pi}{6} < \arg Y < \frac{\pi}{6}. \quad (4.48)$$

In the matching procedure, one also obtains the jump condition

$$a_3^{(1)+} - a_3^{(1)-} = -\mathrm{i}\pi\left[2\frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right) + \frac{16\lambda}{3}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right) + \frac{A}{2\lambda}\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)\right]. \quad (4.49)$$

The first harmonic satisfies the homogeneous equation

$$L^{*(2)}[\Phi_3^{*(2)}] = 0 \quad (4.50)$$

which, after matching yields

$$\Phi_3^{*(2)} = a_3^{(2)} \quad (4.51)$$

and

$$a_3^{(2)+} = a_3^{(2)-} = a_3^{(2)}. \quad (4.52)$$

Finally the second harmonic $\Phi_3^{*(3)}$ is given by

$$L^{*(3)}[\Phi_3^{*(3)}] = -\frac{\pi\lambda^{-\frac{4}{3}}}{24}(1-\mathrm{i}\sqrt{3})\left(\frac{\partial A^3}{\partial T_1}+24\lambda A^3\right)w_1''''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] \quad (4.53)$$

and the matching condition (C 1). One can show that the solution is

$$\Phi_3^{*(3)} = \frac{\pi}{12\lambda}\left(\frac{\partial A^3}{\partial T_1}+24\lambda A^3\right)w_2\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] + 3a_3^{(3)}, \quad (4.54)$$

where $a_3^{(3)}$ is a constant which could be determined by solving the corresponding outer problem (3.29), and $w_2(z)$ is the particular solution of

$$\frac{\mathrm{d}^4 w_2}{\mathrm{d}z^4} - 3z\frac{\mathrm{d}^2 w_2}{\mathrm{d}z^2} = w_1''''(z) \quad (4.55)$$

which admits the asymptotic expansion

$$w_2(z) \sim -\frac{\pi^{-1}}{24z^3}+\dots \quad \text{when} \quad |z| \to \infty; \; -\pi < \arg z < \tfrac{1}{3}\pi. \tag{4.56}$$

$O(\epsilon^{\frac{10}{3}})$ *problem.* Fortunately, we shall only need to examine the fundamental and 1st harmonic part of the solution. The fundamental is governed by the following equation

$$\begin{aligned} L^{*(1)}[\Phi^{*(1)}_{\frac{10}{3}}] = {} & 2\frac{\partial A}{\partial X_1}Y^3+4\mathrm{i}\lambda\frac{\partial A}{\partial X_1}-\tfrac{5}{2}\mathrm{i}a_2^{(1)}Y^3+5\lambda a_2^{(1)}-2\left(\frac{\partial a_2^{(1)}}{\partial T_1}+4\lambda a_2^{(1)}\right)-2\frac{\partial A}{\partial T_2} \\ & -2\mathrm{i}\left(b_2^{(0)}+\frac{1}{2\lambda}\frac{\partial \bar{P}^{(0)}_{2\infty}}{\partial X_1}+\frac{1}{8\lambda}\frac{\partial |A|^2}{\partial X_1}+\frac{1}{2\lambda}\frac{\partial b_2^{(0)}}{\partial T_1}-\frac{1}{2\lambda}\frac{\partial a_2^{(0)}}{\partial X_1}\right)A \\ & -2\pi\mathrm{i}\frac{\partial}{\partial X_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Yw''\left[\left(\frac{\mathrm{i}}{\lambda}\right)^{\frac{1}{3}}Y\right] \end{aligned} \tag{4.57}$$

and the matching condition (C 14). It is easy to determine the asymptotic behaviour of $\Phi^{*(1)}_{\frac{10}{3}}$ as $|Y| \to \infty$ and to deduce its outer expansion. The outcome of the calculations is the jump condition

$$\begin{aligned} b_3^{(1)+}-b_3^{(1)-} = \mathrm{i}\pi\Bigg[& \frac{\partial}{\partial X_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)-\mathrm{i}\frac{\partial A}{\partial T_2}-\mathrm{i}\left(\frac{\partial a_2^{(1)}}{\partial T_1}+4\lambda a_2^{(1)}\right) \\ & +b_2^{(0)}+\frac{1}{2\lambda}\frac{\partial \bar{P}^{(0)}_{2\infty}}{\partial X_1}+\frac{1}{8\lambda}\frac{\partial |A|^2}{\partial X_1}+\frac{1}{2\lambda}\frac{\partial b_2^{(0)}}{\partial T_1}-\frac{1}{2\lambda}\frac{\partial a_2^{(0)}}{\partial X_1}\Bigg] \end{aligned} \tag{4.58}$$

which is the relation that was needed to complete the derivation of the second amplitude equation (3.25).

In order to derive the jump condition pertaining to the 1st harmonic it is sufficient to know its asymptotic behaviour for large $|Y|$. The leading order terms in the forcing term are such that

$$L^{*(2)}[\Phi^{*(2)}_3] \sim 4\mathrm{i}A^2Y^3-4\lambda A^2+2\left(\frac{\partial A^2}{\partial T_1}+14\lambda A^2\right)+O\left(\frac{1}{Y^3}\right) \tag{4.59}$$

subject to the matching condition (C 18). The jump condition then follows as

$$b_3^{(2)+}-b_3^{(2)-} = -\frac{\pi}{3}\left(\frac{\partial A^2}{\partial T_1}+14\lambda A^2\right). \tag{4.60}$$

This completes the inner critical layer analysis. The main results are discussed in the next section.

5. Discussion of the jump conditions

A simple physical interpretation of the relations derived in the previous section is readily obtained by considering the jumps in vorticity or velocity which occur as one crosses the critical layer. For instance, the constants $a_2^{(0)\pm}$, $b_2^{(0)\pm}$, $\partial \bar{P}^{(0)\pm}_{2\infty}/\partial X_1$ arising in the $O(\epsilon^2)$-mean flow distortion given in equation (3.30) have been shown to take the same value above and below the origin. The constants $c_2^{(0)\pm}$, however, are related by equation (4.35) which can be written as

$$\Omega_2^{(0)+}-\Omega_2^{(0)-} = \frac{1}{\lambda}\frac{\partial |A|^2}{\partial X_1}, \tag{5.1}$$

where $\Omega_2^{(0)+}$ and $\Omega_2^{(0)-}$ are the values of the $O(\epsilon^2)$ mean vorticity immediately above and below the critical layer. It may therefore be concluded that the mean velocity is continuous, but the mean vorticity jumps by the amount given in (5.1). It is also interesting to notice that this discontinuity only takes place in the case of spatially growing waves. Otherwise, it is zero.

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In order to discuss the results pertaining to the fundamental, we find it convenient to assume, without loss of generality, that the normalizing constant $a_2^{(0)}$ is identically zero. Relation (4.30) may then be cast in the following form

$$\psi_2^{(1)+} - \psi_2^{(1)-} = (y\,\mathrm{sech}\,y + \sinh y)\,\mathrm{Re}\,\pi(\partial A/\partial T_1 + 4\lambda A)\,\mathrm{e}^{\mathrm{i}\xi}. \tag{5.2}$$

The $O(\epsilon^2)$ axial velocity $u_2^{(1)}$ is discontinuous whereas the transverse velocity $v_2^{(1)}$ is continuous across the origin. Similarly, the jump conditions (4.49) and (4.58) can be written as follows

$$\begin{aligned}\psi_3^{(1)+} - \psi_3^{(1)-} = {}& -\mathrm{sech}\,y\,\mathrm{Re}\,\mathrm{i}\pi\left[2\frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)+\frac{16\lambda}{3}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)+\frac{A}{2\lambda}\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)\right]\mathrm{e}^{\mathrm{i}\xi}\\ &+(y\,\mathrm{sech}\,y+\sinh y)\,\mathrm{Re}\,\mathrm{i}\pi\left[\frac{\partial}{\partial X_1}\left(\frac{\partial A}{\partial T_1}+4\lambda A\right)-\mathrm{i}\frac{\partial A}{\partial T_2}\right.\\ &\left.+\left(b_2^{(0)}+\frac{1}{2\lambda}\frac{\partial \bar{P}_{2\infty}^{(0)}}{\partial X_1}+\frac{1}{8\lambda}\frac{\partial|A|^2}{\partial X_1}+\frac{1}{2\lambda}\frac{\partial b_2^{(0)}}{\partial T_1}-\frac{1}{2\lambda}\frac{\partial a_2^{(0)}}{\partial X_1}\right)A\right]\mathrm{e}^{\mathrm{i}\xi}.\end{aligned} \tag{5.3}$$

Hence the $O(\epsilon^3)$ contribution to the fundamental not only experiences a discontinuity in axial velocity, but also in transverse velocity as evidenced by the presence of the 1st term in (5.3). Finally the axial velocity of the 1st harmonic becomes discontinuous at $O(\epsilon^3)$ as shown by the relation

$$\psi_3^{(2)+} - \psi_3^{(2)-} = -\tfrac{1}{3}\pi(\sinh 2y + \tanh y)\,\mathrm{Re}\,(\partial A^2/\partial T_1 + 14\lambda A^2)\,\mathrm{e}^{\mathrm{i}\xi}. \tag{5.4}$$

The above results can favourably be compared with the work of Haberman (1972) which was mentioned in the Introduction. Bearing in mind that in the present study, the Reynolds number in the critical layer ($1/\lambda_c$, in Haberman's terminology) is $O(\epsilon^{\frac{1}{2}})$ and that the neutral mode becomes singular at $O(\epsilon)$ only, one can easily establish that the $O(\epsilon^2)$ mean vorticity jump of this investigation is perfectly compatible with the $O(\epsilon^{\frac{1}{2}})$ change in mean shear of Haberman's paper. In the same manner, the $O(\epsilon^2)$ jump in axial velocity and $O(\epsilon^3)$ jump in transverse velocity become respectively $O(\epsilon)$ and $O(\epsilon^{\frac{3}{2}})$ in Haberman's case. The necessity of a mean vorticity jump across the critical layer has also been discussed in detail by Stewartson (1978) and Brown & Stewartson (1978) in the context of a weak Rossby wave forced on a uniform shear flow. Brown & Stewartson (1978) in particular, showed how for large λ_c, this jump eventually spreads throughout the outer flow by viscous diffusion. In the present situation, when we restrict ourselves to free spatially growing disturbances, the same phenomenon occurs: the mean vorticity jump acts as a source term in the mean flow distortion equation (3.19), thereby leading to its own diffusion away from the origin under the influence of the viscous stress term $-\lambda\,\partial^3\Phi_2^{(0)}/\partial y^3$. Furthermore, by comparing the jump conditions derived in section 4 with the asymptotic behaviour of the outer solution close to the origin given in appendix C, one may also conclude that, in most cases, it is legitimate to interpret the logarithmic functions arising in the outer expansion by appropriately choosing a branch which yields a $-\pi$ phase shift as the origin is crossed from below. For the mean flow distortion term, however, this 'rule' breaks down and nothing can replace the method of matched asymptotic expansions. The jump in phase shift is a satisfactory concept as long as the outer flow is inviscid and linear, but this is not the case here, as evidenced by the presence of the harmonics and the viscous terms derived in § 3. We have therefore preferred to follow the suggestion of Stewartson (1978) and interpret our results in terms of velocity and vorticity jumps.

6. The amplitude equations and mean flow distortion equation

The results of the inner analysis enable us to complete the determination of the amplitude and mean flow distortion equations formally derived in § 3. The 1st amplitude equation can immediately be deduced from the substitution of (4.30) into (3.14) so as to read

$$\frac{\partial A}{\partial T_1}-\frac{2\mathrm{i}}{\pi}\frac{\partial A}{\partial X_1}+4\lambda A=0. \tag{6.1}$$

In the same fashion relations (4.58) and (3.25) lead to the second amplitude equation

$$\pi\frac{\partial A}{\partial T_2}-2\mathrm{i}\frac{\partial A}{\partial X_2}-24\lambda\frac{\partial A}{\partial T_1}-\frac{8\mathrm{i}\lambda}{\pi}\left(\tfrac{11}{3}-4\chi_3(1)\right)\frac{\partial A}{\partial X_1}-\frac{\partial^2 A}{\partial X_1^2}+\frac{2\mathrm{i}}{\pi}(4\chi_3(1)-3)\frac{\partial^2 A}{\partial T_1\partial X_1}-192\lambda^2 A+\tfrac{4}{3}|A|^2A$$
$$+\frac{A}{2}\int_{-\infty}^{+\infty}\operatorname{sech}^2 y\coth y\left(\frac{\partial^3\Phi_2^{(0)}}{\partial y^3}+2\operatorname{sech}^2 y\frac{\partial\Phi_2^{(0)}}{\partial y}\right)\mathrm{d}y$$
$$+\mathrm{i}\pi\left[\left.\frac{\partial\Phi_2^{(0)}}{\partial y}\right|_0+\frac{1}{2\lambda}\left(\left.\frac{\partial^2\Phi_2^{(0)}}{\partial T_1\partial y}\right|_0-\left.\frac{\partial\Phi_2^{(0)}}{\partial X_1}\right|_0+\frac{\partial\bar{P}_{2\infty}^{(0)}}{\partial X_1}+\frac{1}{4}\frac{\partial|A|^2}{\partial X_1}\right)\right]A=0. \tag{6.2}$$

In the above relation, the integral arising from coupling with the mean flow must be interpreted in the manner described in (3.27) without taking into account singularities which may be present at the origin. Furthermore, the constants $b_2^{(0)}$ and $a_2^{(0)}$ have been replaced by the equivalent expressions $\partial\Phi_2^{(0)}/\partial y|_0$ and $\Phi_2^{(0)}|_0$. Finally, the pressure gradient at infinity has been shown to take identical values on both sides of the critical layer so that the mean flow distortion equation can be written as

$$\left(\frac{\partial}{\partial T_1}+\tanh y\frac{\partial}{\partial X_1}\right)\frac{\partial\Phi_2^{(0)}}{\partial y}-\operatorname{sech}^2 y\frac{\partial\Phi_2^{(0)}}{\partial X_1}-\lambda\frac{\partial^3\Phi_2^{(0)}}{\partial y^3}$$
$$=-\tfrac{1}{2}\operatorname{sech}^4 y\coth y\left(\frac{\partial|A|^2}{\partial T_1}+8\lambda|A|^2\right)-\tfrac{1}{4}\operatorname{sech}^2 y\frac{\partial|A|^2}{\partial X_1}+12\lambda\operatorname{sech}^4 y\tanh y\,|A|^2-\frac{\partial\bar{P}_{2\infty}^{(0)}}{\partial X_1}. \tag{6.3}$$

where the vorticity $-\partial^2\Phi_2^{(0)}/\partial y^2$ must satisfy the jump condition

$$\left.\frac{\partial^2\Phi_2^{(0)}}{\partial y^2}\right|_{0^-}^{0^+}=-\frac{1}{\lambda}\frac{\partial|A|^2}{\partial X_1}. \tag{6.4}$$

We consider relations (6.1), (6.2), (6.3) and (6.4) to be the main results of this paper. When the viscous parameter λ is set equal to zero and one assumes that the amplitude is only a function of X_2 and T_2, the amplitude equation of Benney & Maslowe (1975) is recovered in the form

$$\frac{\partial A}{\partial T_2}-\frac{2\mathrm{i}}{\pi}\frac{\partial A}{\partial X_2}+\frac{4}{3\pi}|A|^2A=0 \tag{6.5}$$

which for temporally growing waves reduces to the equations obtained by Schade (1964) and Huerre (1977). For reasons already stated in the introduction, it is felt that this simpler equation does not fully describe the finite amplitude regime of weakly amplified waves in the viscous critical layer regime since the corresponding transverse distribution of the fluctuations cannot possibly be matched across the origin by a *viscous* critical layer. We shall not attempt to find a general solution of this set of equations but rather examine a few particular situations which will illustrate their most important features. For obvious physical reasons, we will consider the more realistic shear layer

$$\bar{U}(y)=U+\tanh y, \tag{6.6}$$

where U is an imposed constant velocity which may take any appropriate value larger than unity. The previous relations can then be adapted to this profile by performing the translation $\xi = x - Ut$ where x is the new longitudinal coordinate. Similar transformations are applied to the slow space scales X_i and T_i. For convenience, we shall keep the same notation and write

$$\frac{\partial A}{\partial T_1} + \left(U - \frac{2\mathrm{i}}{\pi}\right)\frac{\partial A}{\partial X_1} + 4\lambda A = 0. \tag{6.7}$$

In the case of temporally growing waves, we assume that the wavenumber is given and differs from the inviscid neutral wavenumber by a small amount $\beta\epsilon$

$$K = 1 - \beta\epsilon. \tag{6.8}$$

The first order fluctuation is then cast in the following form

$$\psi_1 = \operatorname{sech} y \operatorname{Re} a(T_1, X_2, T_2)\, \mathrm{e}^{\mathrm{i}(1-\beta\epsilon)(x-Ut)}, \tag{6.9}$$

where the amplitude function $a(T_1, X_2, T_2)$ is related to the initial $A(X_i, T_i)$ by

$$A(X_i, T_i) = a(T_1, X_2, T_2)\, \mathrm{e}^{-\mathrm{i}\beta(X_1 - UT_1)}. \tag{6.10}$$

Substitution of the above expression into the first amplitude equation (6.7) yields

$$\frac{\partial a}{\partial T_1} + \frac{2(2\lambda\pi - \beta)}{\pi} a = 0. \tag{6.11}$$

Hence, to leading order in ϵ, the temporal amplification rate is

$$S_i = \frac{2(\beta - 2\lambda\pi)}{\pi}\epsilon + O(\epsilon^2) \tag{6.12}$$

or, in terms of the wavenumber K and Reynolds number R,

$$S_i = \frac{2}{\pi}\left(1 - \frac{2\pi}{R} - K\right) + O[(1-K)^2]. \tag{6.13}$$

The linear amplitude equation (6.7) is therefore solely associated with finite Reynolds number effects. Whereas the inviscid neutral wave number is unity, the true neutral wavenumber

$$K_n = 1 - 2\pi/R + O(R^{-2}) \tag{6.14}$$

has been shifted towards lower wavenumbers. This is entirely consistent with the numerical results obtained by Betchov & Szewszyk (1963) and reproduced in figure 1. The slope of the amplification rate curve is however unchanged at this order.

In the case of spatially growing waves, similar results can be obtained by suitably choosing the dependence of A on T_1 so that the nondimensional frequency S is real and within $O(\epsilon)$ of the inviscid neutral frequency U. It is then found that the real part K_r of the wavenumber and the spatial amplification rate K_i are respectively given by

$$K_r = \frac{1}{U^2 + 4/\pi^2}\left[\frac{4}{\pi^2}\left(1 - \frac{2\pi}{R}\right) + US\right] + O(U-S)^2, \tag{6.15}$$

$$K_i = -\frac{2U}{\pi}\,\frac{1 - 2\pi/R - S/U}{U^2 + 4/\pi^2} + O(U-S)^2. \tag{6.16}$$

The true neutral frequency S_n experiences the same downward shift as in the temporal case

$$S_n = UK_n = U(1-2\pi/R)+O(R^{-2}). \tag{6.17}$$

From the above comments, we conclude that the 1st amplitude equation accounts for the fact that the Reynolds number is not infinite but $O(\epsilon^{-1})$.

We are ultimately interested in wavenumbers and frequencies within $O(\epsilon^2)$ of the true neutral wavenumber or frequency, and in the ensuing discussion of the mean flow distortion equation and second amplitude equation, we choose the solution of (6.7) which is associated with the neutral wavenumber K_n and such that

$$\psi_1 = \operatorname{sech} y \operatorname{Re} b(X_2, T_2)\, e^{-2i\lambda\pi(X_1-UT_1)} \tag{6.18}$$

or

$$A(X_i, T_i) = b(X_2, T_2)\, e^{-2i\lambda\pi(X_1-UT_1)}. \tag{6.19}$$

Since, in this case, the fluctuations are neutral with respect to their dependence on X_1 and T_1, the mean flow vorticity is continuous at the origin (see equation (6.4)), and the mean flow distortion equation (6.3) reduces to

$$\frac{\partial^3 \Phi_2^{(0)}}{\partial y^3} = 4\operatorname{sech}^4 y(\coth y - 3\tanh y)\,|b(X_2,T_2)|^2 + \frac{1}{\lambda}\frac{\partial \overline{P}_{2\infty}^{(0)}}{\partial X_1}. \tag{6.20}$$

The corresponding longitudinal velocity is then found to be

$$\frac{\partial \Phi_2^{(0)}}{\partial y} = [4y\ln|\tanh y| + \operatorname{sech}^2 y\tanh y + 4\tanh y - 4\chi_2(\tanh y)]\,|b(X_2,T_2)|^2 + \frac{1}{2\lambda}\frac{\partial \overline{P}_{2\infty}^{(0)}}{\partial X_1}y^2 + \gamma y + \alpha, \tag{6.21}$$

and α and γ are arbitrary functions of the slow variables which take identical values on both sides of the critical layer. These quantities are to be determined by imposing appropriate boundary conditions at $y = \pm\infty$. The basic mean velocity $\tanh y$ is bounded as y goes to infinity, and in order to preserve the validity of the outer expansion for large y, it is legitimate to require that the mean distortion velocity be also bounded. It will, therefore, be assumed that γ is identically zero, i.e. that the mean distortion vorticity is zero at infinity. In order to specify $\partial\Phi_2^{(0)}/\partial y$ uniquely, we will further assume that the mean pressure gradient vanishes at infinity and that the net mean mass flux is not changed by the introduction of the fluctuations. Under these assumptions, the axial velocity is given by

$$\partial\Phi_2^{(0)}/\partial y = [4y\ln|\tanh y| + \operatorname{sech}^2 y\tanh y + 4\tanh y - 4\chi_2(\tanh y)]/b(X_2,T_2)|^2, \tag{6.22}$$

and the associated stream function is

$$\Phi_2^{(0)} = \left[2\int_1^{\tanh y}\frac{(\operatorname{artanh} t)^2}{t}\,dt + 2y^2\ln|\tanh y| - 4y\chi_2(\tanh y) + 4\ln(2\cosh y) - \tfrac{1}{2}\operatorname{sech}^2 y\right]|b(X_2,T_2)|^2, \tag{6.23}$$

One may notice that, in this instance, the requirement of zero longitudinal velocity at infinity is incompatible with the mean flow distortion equation.

The form of the distorted mean flow profile is displayed in figure 2. It can be compared with the corresponding result of Maslowe (1977) for finite Reynolds numbers. Whereas Maslowe's expression for the mean distortion behaves like R^{-1} for large Reynolds numbers, it is, in the context of the present study, independent of R as $R \to \infty$. Bearing in mind the choice of boundary conditions made earlier, the velocity is zero at the origin and stays constant at infinity. One also notes that the infinite slope at the origin is due to the logarithmic singularity in the outer expansion. An expression which is valid in the critical layer is given by equation (4.34). Furthermore, the mean velocity distortion is antisymmetric as long as the fundamental wavenumber is within $O(\epsilon^2)$ of the neutral wavenumber (6.14). If the wave is allowed to grow in time or space according to (6.13) or (6.16), the mean flow distortion equation (6.3) does not reduce to the simple form (6.21) and one obtains a much more complicated distortion of the mean flow.

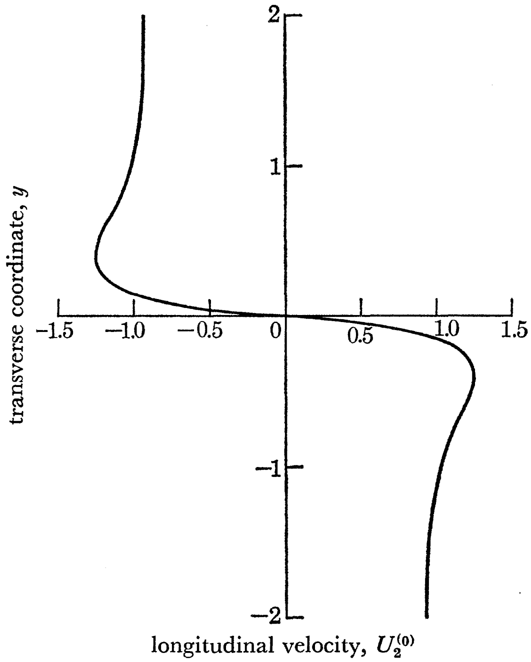

FIGURE 2. Mean flow distortion.

Having restricted the form of the amplitude function to (6.19) and calculated the function $\Phi_2^{(0)}$, it is straightforward to derive the evolution equation pertaining to $b(X_2, T_2)$ from the general equation (6.2). One finds that $b(X_2, T_2)$ is governed by

$$\frac{\partial b}{\partial T_2}+\left(U-\frac{2\mathrm{i}}{\pi}\right)\frac{\partial b}{\partial X_2}-\frac{16\lambda^2}{\pi}\left[\frac{47}{3}-\frac{\pi^2}{4}-4\chi_3(1)\right]b-\frac{16}{\pi}\left[\tfrac{1}{3}+\chi_3(1)\right]|b^2|\,b=0, \tag{6.24}$$

where $\chi_3(1)$ is given by (3.26) or equivalently

$$\chi_3(1)=\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3}=1.0518. \tag{6.25}$$

After a suitable U translation along the x axis, this equation can be directly compared with Benney & Maslowe's result (6.5). In this earlier investigation, the third linear term in b is absent. As we shall see shortly, it corresponds to the $O(R^{-2})$ change in neutral wavenumber. The crucial nonlinear term determining the Landau constant has changed sign, and this is solely due to the effect of the mean flow distortion coupling term in (6.2). The evolution equations pertaining to time or space growing waves can be obtained from (6.24) by following the same procedure as for the first amplitude equation.

For temporally growing waves, the linear neutral wavenumber of (6.14) can now be expanded to $O(R^{-2})$ to read

$$K_n = 1 - \frac{2\pi}{R} + \frac{8}{R^2}\left(\frac{47}{3} - \frac{\pi^2}{4} - 4\chi_3(1)\right) + O\left(\frac{1}{R^3}\right). \tag{6.26}$$

When the real wavenumber K is within $O(\epsilon^2)$ of K_n and such that

$$K = K_n - \Delta K \epsilon^2, \tag{6.27}$$

the temporal amplitude $c(T_2)$ defined by

$$\psi_1 = \operatorname{sech} y \operatorname{Re} c(T_2) \exp\{\mathrm{i}(K_n - \Delta K\epsilon^2)(x - Ut)\} \tag{6.28}$$

satisfies the following evolution equation

$$\mathrm{d}c/\mathrm{d}T_2 = (2/\pi)\left[\Delta K + 8(\tfrac{1}{3} + \chi_3(1))\,|c|^2\right] c. \tag{6.29}$$

Below the neutral wave number $\Delta K > 0$, there is no equilibrium amplitude since the non-linear term is destabilizing. Above the neutral wavenumber $\Delta K < 0$, a threshold amplitude exists

$$|c| = \left[\frac{|\Delta K|}{8(\frac{1}{3} + \chi_3(1))}\right]^{\frac{1}{2}}. \tag{6.30}$$

Oscillations will grow in amplitude above this threshold and decay below.

Similarly, in the case of spatially growing waves, the linear neutral frequency becomes to $O(\epsilon^2)$

$$S_n = U\left[1 - \frac{2\pi}{R} + \frac{8}{R^2}\left(\frac{47}{3} - \frac{\pi^2}{4} - 4\chi_3(1)\right)\right] + O\left(\frac{1}{R^3}\right), \tag{6.31}$$

and the spatial amplitude function $f(X_2)$ associated with the frequency

$$S = S_n - \Delta S \epsilon^2 \tag{6.32}$$

and defined by

$$\psi_1 = \operatorname{sech} y \operatorname{Re} f(X_2) \exp\{\mathrm{i}K_n(x - Ut) + \mathrm{i}\Delta S\epsilon^2 t\} \tag{6.33}$$

obeys the amplitude equation

$$\frac{\mathrm{d}f}{\mathrm{d}X_2} = \frac{1}{U^2 + 4/\pi^2}\left[\left(\frac{2}{\pi} - \mathrm{i}U\right)\Delta S + \frac{16}{\pi}\left(U + \frac{2\mathrm{i}}{\pi}\right)(\tfrac{1}{3} + \chi_3(1))\,|f|^2\right] f. \tag{6.34}$$

The imaginary terms on the right-hand side of (6.34) correspond to a change in phase speed of the wave. The magnitude of the complex amplitude f satisfies the equation

$$\frac{\mathrm{d}\,|f|}{\mathrm{d}X_2} = \frac{2}{\pi(U^2 + 4/\pi^2)}\left[\Delta S + 8U(\tfrac{1}{3} + \chi_3(1))\,|f|^2\right] |f|. \tag{6.35}$$

Here again there is no equilibrium amplitude but a threshold amplitude exists above the neutral frequency.

It can, therefore, be concluded that the mean flow distortion contribution to the Landau constant is positive and larger in magnitude than the 1st harmonic negative contribution. Hence, the total Landau constant is positive and leads to further amplification of linearly unstable temporal or spatial waves.

It should be emphasized that this conclusion does not depend on the particular choice of boundary conditions made earlier in order to specify the mean flow distortion uniquely: the general solution of the mean flow distortion equation, as given by (6.21), depends on three

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arbitrary functions $\partial \bar{P}_{2\infty}^{(0)}/\partial X_1$, γ and α. It can be readily checked that a nonzero pressure gradient at infinity or a nonzero value of α would not affect the mean flow coupling term in the amplitude equation (6.2). The only term which might contribute to the mean flow coupling term arises from the arbitrary function γ, but it was shown that γ must be zero if the longitudinal mean flow distortion velocity is to be finite for large y.

The result of the present investigation sharply differs from previous studies and in particular from earlier work by the author. In all these instances, however, the mean flow distortion was effectively assumed to be zero. At first sight, we also seem to disagree with the comments of Stuart (1960) regarding the sign of the part k_1 of the Landau constant which arises from the distortion of the mean motion. Stuart showed that k_1 is always negative. Indeed, by specializing his equation (6.3) to the situation which concerns us here, k_1 takes the form

$$k_1 = -\frac{\epsilon}{16\lambda k_0 |A|^4}\int_{-\infty}^{+\infty} [\Phi_1^{(1)\prime}\tilde{\Phi}_2^{(1)} - \tilde{\Phi}_1^{(1)\prime}\Phi_2^{(1)} + \Phi_2^{(1)\prime}\tilde{\Phi}_1^{(1)} - \tilde{\Phi}_2^{(1)\prime}\Phi_1^{(1)}]^2\,\mathrm{d}y, \tag{6.36}$$

where the tilde denotes the complex conjugate. $\Phi_1^{(1)}$ is the fundamental eigenfunction $A\,\mathrm{sech}\,y$ and k_0 is the energy of the fundamental wave

$$k_0 = \frac{1}{4|A|^2}\int_{-\infty}^{+\infty} [|\Phi_1^{(1)\prime}|^2 + |\Phi_1^{(1)}|^2]\,\mathrm{d}y. \tag{6.37}$$

The above expressions are obtained by applying the energy method, and in this context, k_1 represents the work done by the $O(\epsilon^3)$ Reynolds stresses on the mean flow distortion $\Phi_2^{(0)}$. Moreover, since k_1 is $O(\epsilon)$, it will only appear in the 3rd amplitude equation governing the variations of A with X_3 and T_3. In fact, the mean flow distortion terms in the second amplitude equation are not associated with k_1, but, in Stuart's terminology, with the part k_3 of the Landau constant arising from the distortion of the fundamental. More specifically, the $O(\epsilon^2)$ mean flow change induces an $O(\epsilon^3)$ distortion of the fundamental and the work done by the corresponding $O(\epsilon^4)$ Reynolds stresses on the basic flow $\bar{U}(y)$ is included in k_3. According to Stuart, the sign of k_3 cannot be determined a priori. The present study shows that k_3 is positive for a hyperbolic tangent free shear layer at high values of R and low values of $R_{\mathrm{c.l.}}$. In his numerical study of the finite Reynolds number case, Maslowe (1977) indicated that the part of the Landau constant pertaining to the harmonic was reduced by 43 % from its inviscid value when $R = 40$. It could also be of interest to study the variations of k_1 and k_3 as the Reynolds number decreases. One may note that, at finite Reynolds number k_1 will become of order unity and will therefore have a stabilizing effect.

7. Concluding remarks

Even though we have been concerned with shear layer Reynolds numbers of $O(1/\epsilon)$, it has been necessary to assume that the shear layer is parallel. For this purpose the $O(\epsilon)$ diffusive effect of viscosity on the basic flow has been effectively neglected by introducing a suitable body force. Such a difficulty would not have been encountered at large values of $R_{\mathrm{c.l.}}$, since, in this instance the shear layer Reynolds number R can be made large enough to keep the basic flow parallel to $O(\epsilon^3)$. The problem of mean flow growth has also been raised by Maslowe (1977) in connection with his estimate of mean flow distortion at finite Reynolds numbers. When R is large and given by (2.4), a proper basic flow defined by the stream function $\bar{\Psi}(y, X_1)$ should obey the boundary layer equation

$$\left(\frac{\partial\bar{\Psi}}{\partial y}\frac{\partial}{\partial X_1} - \frac{\partial\bar{\Psi}}{\partial X_1}\frac{\partial}{\partial y}\right)\frac{\partial\bar{\Psi}}{\partial y} - \lambda\frac{\partial^3\bar{\Psi}}{\partial y^3} = 0 \tag{7.1}$$

and it has been shown by Lock (1951) that this equation admits a family of slowly growing free shear layer solutions of the form $\bar{U}(y/\sqrt{X_1})$. In principle one can then use the same method as Crighton & Gaster (1976) and seek a solution of the form

$$\psi_1 = \operatorname{Re} A(X_i, T_i)\, \Phi_1^{(1)}\left(\frac{y}{\sqrt{X_1}}\right) \mathrm{e}^{\mathrm{i}\Theta(X_1, T_1)/\epsilon}, \tag{7.2}$$

where Θ is a phase function and the local frequency and wavenumber are defined in the usual way. In the linear problem, this formulation is very convenient. However, for the weakly nonlinear stability approach to succeed, one has to choose among the possible solutions of the linear problem a wavenumber-frequency pair which is neutral for all X_1. This is in general impossible to achieve since a neutral solution at one station X_1 will be decaying further downstream and growing further upstream. Hence there is an inherent difficulty in applying the weakly nonlinear theory to slowly diverging flows.

Keeping in mind the parallel flow assumption, one can then state the main result of this study in the following way: at low critical layer Reynolds numbers and high shear layer Reynolds numbers, the change in the mean flow transverse distribution generated by Reynolds stresses is such that weakly amplified waves do not reach an equilibrium amplitude as they evolve in time or along the downstream direction. Correspondingly, linearly damped waves become unstable if their amplitude exceeds a threshold level.

When compared with earlier work, this conclusion should not be viewed as resulting from the introduction of an artificial body force. As stated in the introduction, a body force was also implicitly present in all previous investigations, and the sign reversal of the Landau constant from a negative to a positive value is solely due to the effect of the mean flow distortion. It must also be emphasized that this result does not exclude the ultimate finite-amplitude equilibration of the instability wave which is known to occur in experiments. It merely states that, for sufficiently small amplitudes, the waves are not stabilized by weakly nonlinear interactions. However, as the wave amplitude increases, the critical layer Reynolds number $R_{\mathrm{c.l.}}$ which is initially small also increases so that nonlinear effects become more important in the critical layer. As described in a later study by Huerre & Scott (1980), this change in critical layer structure may in turn lead to a decrease in the amplification rate until the wave reaches an equilibrium amplitude. Indeed, Miksad (1972) showed that, in the transition of free shear layers, $R_{\mathrm{c.l.}}$ increases from 10^{-3} to 10 as the instability wave grows along the shear layer and eventually reaches an equilibrium amplitude.

Hence, the equilibration stage only takes place at relatively large values of $R_{\mathrm{c.l.}}$. As far as the amplification of tailpipe disturbances in a fully turbulent jet is concerned, one may also expect $R_{\mathrm{c.l.}}$ to increase like the $\frac{3}{2}$ power of the fluctuation level. Absolute values of $R_{\mathrm{c.l.}}$ are likely to be small since the effective viscosity is considerably larger than in the associated laminar flow. In this context, it would be of interest to determine if the mean flow distortion is destabilizing for two-dimensional or axisymmetric jet profiles, as it is for free shear layers.

This work also seems to imply that the regular inviscid equilibrium solutions of Stuart (1967) do not result from the evolution of linearly amplified waves in the viscous critical layer régime. This further supports the statement of Maslowe (1977) indicating that Stuart's steady solutions arise from the evolution of finite-amplitude waves in the nonlinear critical layer régime.

It is very likely that within the parallel flow approximation the formulation presented in this paper can be generalized to finite values of the Reynolds number in the critical layer. Such a study will provide valuable information concerning the evolution of finite amplitude waves at high fluctuation levels. It will also lead to a detailed description of the critical layer for nearly neutral waves when both viscosity and nonlinearities have to be taken into account.

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Appendix A

In this appendix, the mean flow correction equation is derived directly from the Navier Stokes equations, in a manner which, it is hoped, will shed some light on its physical meaning. The approach is very similar to the original energy method of Stuart (1958) which is formulated here within the framework of the method of multiple scales.

Let $\tilde{p}$, $\tilde{u}$ and $\tilde{v}$ be the pressure and the ξ and y components of the velocity nondimensionalized with respect to the length scale and velocity scales defined in § 2. The Navier Stokes equations then read as follows:

$$\frac{\partial \tilde{u}}{\partial \xi}+\frac{\partial \tilde{v}}{\partial y}=0, \tag{A 1}$$

$$\frac{\partial \tilde{u}}{\partial t}+\tilde{u}\frac{\partial \tilde{u}}{\partial \xi}+\tilde{v}\frac{\partial \tilde{u}}{\partial y}=-\frac{\partial \tilde{p}}{\partial \xi}+\lambda\epsilon\nabla^2\tilde{u}+f, \tag{A 2}$$

$$\frac{\partial \tilde{v}}{\partial t}+\tilde{u}\frac{\partial \tilde{v}}{\partial \xi}+\tilde{v}\frac{\partial \tilde{v}}{\partial y}=-\frac{\partial \tilde{p}}{\partial y}+\lambda\epsilon\nabla^2\tilde{v}, \tag{A 3}$$

where f is the body force introduced in the first order outer problem and defined by equation (3.3). As in the rest of the study, the dependent variables may be expanded in powers of ϵ in the following way

$$\tilde{u}=\bar{U}(y)+\epsilon u_1^{(1)}(\xi,y,X_i,T_i)+\epsilon^2[\bar{U}_2^{(0)}(y,X_i,T_i)+u_2^{(1)}(\xi,y,X_i,T_i)+u_2^{(2)}(\xi,y,X_i,T_i)]+\dots \tag{A 4}$$

$$\tilde{v}=\epsilon v_1^{(1)}(\xi,y,X_i,T_i)+\epsilon^2[v_2^{(1)}(\xi,y,X_i,T_i)+v_2^{(2)}(\xi,y,X_i,T_i)]+\epsilon^3[\bar{V}_3^{(0)}(y,X_i,T_i)+\dots]+\dots \tag{A 5}$$

$$\tilde{p}=\bar{P}+\epsilon p_1^{(1)}(\xi,y,X_i,T_i)+\epsilon^2[\bar{P}_2^{(0)}(y,X_i,T_i)+p_2^{(1)}(\xi,y,X_i,T_i')+p_2^{(2)}(\xi,y,X_i,T_i)]+\dots. \tag{A 6}$$

In the above expressions, the (0), (1), (2) and (3) superscripts characterize the mean flow change, the fundamental, first harmonic and second harmonic fluctuations respectively. Multiple scales have also been introduced so that $\partial/\partial t$ and $\partial/\partial\xi$ admit expansions in powers of ϵ as given by equation (2.7). After substitution into the Navier Stokes equations and averaging over a wavelength of the neutral wave, it is easy to show that the leading order mean flow correction equations are

$$\frac{\partial \bar{U}_2^{(0)}}{\partial T_1}+\bar{U}(y)\frac{\partial \bar{U}_2^{(0)}}{\partial X_1}+\bar{U}'(y)\bar{V}_3^{(0)}=\lambda\frac{\partial^2\bar{U}_2^{(0)}}{\partial y^2}-\frac{\partial \bar{P}_2^{(0)}}{\partial X_1}-\frac{\partial}{\partial y}\overline{(u_1^{(1)}v_2^{(1)}+u_2^{(1)}v_1^{(1)})}-\frac{\partial}{\partial X_1}\overline{(u_1^{(1)})^2} \tag{A 7}$$

$$\frac{\partial \bar{P}_2^{(0)}}{\partial y}=-\frac{\partial}{\partial y}\overline{(v_1^{(1)})^2}. \tag{A 8}$$

The interpretation of these equations is well established in the literature on the subject: the mean flow change is determined by the action of viscous stresses, Reynolds stresses and mean pressure gradient. The second equation of motion (A 8) may immediately be integrated with respect to y, and the change in mean pressure is given by

$$\bar{P}_2^{(0)}(y,X_i,T_i)=-\overline{(v_1^{(1)})^2}+\bar{P}_{2\infty}^{(0)\pm}(X_i,T_i), \tag{A 9}$$

where $\bar{P}_{2\infty}^{(0)\pm}(X_i,T_i)$ is the mean pressure change at infinity, the + and − signs pertaining to $y=+\infty$ and $y=-\infty$ respectively. Substitution into the first equation of motion (A 7) then yields the single relation

$$\frac{\partial \bar{U}_2^{(0)}}{\partial T_1}+\bar{U}(y)\frac{\partial \bar{U}_2^{(0)}}{\partial X_1}+\bar{U}'(y)\bar{V}_3^{(0)}=\lambda\frac{\partial^2\bar{U}_2^{(0)}}{\partial y^2}-\frac{\partial \bar{P}_{2\infty}^{(0)\pm}}{\partial X_1}-\frac{\partial}{\partial y}\overline{(u_1^{(1)}v_2^{(1)}+u_2^{(1)}v_1^{(1)})}-\frac{\partial}{\partial X_1}[\overline{(u_1^{(1)})^2}-\overline{(v_1^{(1)})^2}] \tag{A 10}$$

which is identical to equation (3.18). This procedure, however, allows us to identify unambiguously the constant $\alpha(X_i, T_i)$ of equation (3.18) with the mean pressure gradient at infinity $\partial \bar{P}^{(0)\pm}_{2\infty}/\partial X_1$.

Appendix B

The forcing terms pertaining to $\Phi_3^{(1)}$ and $\Phi_3^{(2)}$ are

$$
\begin{aligned}
Q_3^{(1)}(y, X_i, T_i) = {} & 2\{\operatorname{sech}^3 y \coth y - (y \operatorname{sech} y + \sinh y) \ln|\tanh y| + \operatorname{sech} y\, \chi_2(\tanh y)\} \\
& \times \frac{\partial}{\partial X_1}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) - \operatorname{sech} y (1 + 2\sinh^2 y) \frac{\partial^2 A}{\partial X_1^2} \\
& - 12\lambda \operatorname{sech} y \tanh y (1 + 4\operatorname{sech}^2 y) \frac{\partial A}{\partial X_1} + 2\operatorname{sech}^2 y \coth y \{\operatorname{sech} y \coth y \\
& - (y \operatorname{sech} y + \sinh y) \ln|\tanh y| + \operatorname{sech} y\, \chi_2(\tanh y)\} \frac{\partial}{\partial T_1}\left(\frac{\partial A}{\partial T_1} + 4\lambda A\right) \\
& - 2\operatorname{sech} y \tanh y \frac{\partial^2 A}{\partial T_1 \partial X_1} - 36\lambda \operatorname{sech}^3 y \frac{\partial A}{\partial T_1} - 4\lambda \operatorname{sech}^2 y \{\operatorname{sech}^3 y \coth^4 y \\
& + 2\coth y (1 - 3\tanh^2 y) [(y \operatorname{sech} y + \sinh y) \ln|\tanh y| - \operatorname{sech} y\, \chi_2(\tanh y)] \\
& + 4\cosh y \ln|\tanh y| + 6\operatorname{sech} y\} \left\{\frac{\partial A}{\partial T_1} + 4\lambda A\right\} + 144\lambda^2 \operatorname{sech}^3 y (5\tanh^2 y - 3) A \\
& - 2\mathrm{i} \operatorname{sech} y \frac{\partial A}{\partial X_2} - 2\mathrm{i} \operatorname{sech}^3 y \coth y \frac{\partial A}{\partial T_2} + \operatorname{sech} y \coth y \left(\frac{\partial^3 \Phi_2^{(0)}}{\partial y^3} + 2\operatorname{sech}^2 y \frac{\partial \Phi_2^{(0)}}{\partial y}\right) A \\
& + \tfrac{5}{2} \operatorname{sech}^5 y |A|^2 A - 8\mathrm{i}\lambda \operatorname{sech}^3 y \coth y [(1 - 3\tanh^2 y)\, a_2^{(1)\pm} + \{y(1 - 3\tanh^2 y) \\
& + 3\tanh y\}\, b_2^{(1)\pm}] - 2\mathrm{i}\left[\operatorname{sech} y \frac{\partial a_2^{(1)\pm}}{\partial X_1} + (y \operatorname{sech} y + \sinh y) \frac{\partial b_2^{(1)\pm}}{\partial X_1}\right] \\
& - 2\mathrm{i} \operatorname{sech}^2 y \coth y \left[\operatorname{sech} y \frac{\partial a_2^{(1)\pm}}{\partial T_1} + (y \operatorname{sech} y + \sinh y) \frac{\partial b_2^{(1)\pm}}{\partial T_1}\right]
\end{aligned}
\tag{B 1}
$$

$$
\begin{aligned}
Q_3^{(2)}(y, X_i, T_i) = {} & \tfrac{1}{4}\mathrm{i} \operatorname{sech}^3 y [4\operatorname{sech} y \coth y + \operatorname{sech} y \coth^3 y - 4(y \operatorname{sech} y + \sinh y) \ln|\tanh y| \\
& + 4\operatorname{sech} y\, \chi_2(\tanh y)] \left(\frac{\partial A^2}{\partial T_1} + 8\lambda A^2\right) + \mathrm{i} \operatorname{sech}^4 y \frac{\partial A^2}{\partial X_1} - 45\mathrm{i}\lambda \operatorname{sech}^4 y \tanh y A^2 \\
& + 6\mathrm{i}\lambda \operatorname{sech}^4 y \coth y A^2 + 2a_2^{(1)\pm} A \operatorname{sech}^4 y + 2b_2^{(1)\pm} A \operatorname{sech}^2 y (y \operatorname{sech}^2 y + \tanh y).
\end{aligned}
\tag{B 2}
$$

Appendix C

In this appendix are tabulated the inner expansions of the outer expansions which have been calculated in § 3. For convenience, the outer stream function has been split into four parts, namely, the mean flow, the fundamental, the first and second harmonic. The corresponding inner expansions rewritten in terms of the inner variable Y are presented in tables 1, 2 and 3. The nth row of each table displays the $O(\epsilon^{n/3})$ inner expansion of the $O(\epsilon^{n/3})$ outer expansion. The results pertaining to the second harmonic are very simple and reduce to

$$
O(\epsilon^{n/3}) \text{ inner}\left[O(\epsilon^{n/3}) \text{ outer 2nd harmonic}\right] = \begin{Bmatrix} 0 & \text{when} & n \leqslant 8 \\ 3a_3^{(3)} & \text{when} & n = 9 \end{Bmatrix}. \tag{C 1}
$$

TABLE 1

n	$O(\epsilon^{\frac{1}{3}n})$ inner $[O(\epsilon^{\frac{1}{3}n})$ outer mean flow$]$	
2	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}$	(C2)
3	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}$	
4	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}$	(C3)
5	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}$	
6	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}+(a_2^{(0)\pm}+\frac{1}{45}Y^6)\,\epsilon^2$	(C4)
7	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}+(a_2^{(0)\pm}+\frac{1}{45}Y^6)\,\epsilon^2+b_2^{(0)\pm}\,Y\epsilon^{\frac{7}{3}}$	(C6)
8	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}+(a_2^{(0)\pm}+\frac{1}{45}Y^6)\,\epsilon^2+b_2^{(0)\pm}\,Y\epsilon^{\frac{7}{3}}$ $+\Big\{\frac{17}{2520}Y^8+\Big[c_2^{(0)\pm}+\frac{1}{4\lambda}\Big(\frac{\partial\lvert A\rvert^2}{\partial T_1}+8\lambda\lvert A\rvert^2\Big)(\ln\lvert Y\rvert+\frac{1}{3}\ln\epsilon)\Big]\,Y^2\Big\}\,\epsilon^{\frac{8}{3}}$	(C6)
9	$\frac{1}{2}Y^2\epsilon^{\frac{2}{3}}-\frac{1}{12}Y^4\epsilon^{\frac{4}{3}}+(a_2^{(0)\pm}+\frac{1}{45}Y^6)\,\epsilon^2+b_2^{(0)\pm}\,Y\epsilon^{\frac{7}{3}}$ $+\Big\{\frac{17}{2520}Y^8+\Big[c_2^{(0)\pm}+\frac{1}{4\lambda}\Big(\frac{\partial\lvert A\rvert^2}{\partial T_1}+8\lambda\lvert A\rvert^2\Big)(\ln\lvert Y\rvert+\frac{1}{3}\ln\epsilon)\Big]\,Y^2\Big\}\,\epsilon^{\frac{8}{3}}$ $-\frac{1}{3\pi}\frac{\partial\lvert A\rvert^2}{\partial X_1}\frac{1}{Y}\epsilon^{\frac{8}{3}}+\Big\{a_3^{(0)\pm}+\frac{1}{6\lambda}\Big(\frac{\partial\bar{P}_2^{(0)\pm}}{\partial X_1}+\frac{1}{4}\frac{\partial\lvert A\rvert^2}{\partial X_1}+\frac{\partial b_2^{(0)\pm}}{\partial T_1}-\frac{\partial a_2^{(0)\pm}}{\partial X_1}\Big)\,Y^3\Big\}\,\epsilon^3$	(C7)

TABLE 2

n	$O(\epsilon^{\frac{1}{3}n})$ inner $[O(\epsilon^{\frac{1}{3}n})$ outer fundamental$]$	
3	$A\epsilon$	(C8)
4	$A\epsilon$	
5	$A\epsilon-\frac{1}{2}AY^2\epsilon^{\frac{5}{3}}$	(C9)
6	$A\epsilon-\frac{1}{2}AY^2\epsilon^{\frac{5}{3}}+a_2^{(1)\pm}\epsilon^2$	(C10)
7	$A\epsilon-\frac{1}{2}AY^2\epsilon^{\frac{5}{3}}+a_2^{(1)\pm}\epsilon^2$ $+\Big\{\frac{5}{24}AY^4+\Big[2b_2^{(1)\pm}-6\mathrm{i}\lambda A+\mathrm{i}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)-2\mathrm{i}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)(\ln\lvert Y\rvert+\frac{1}{3}\ln\epsilon)\Big]\,Y\Big\}\,\epsilon^{\frac{7}{3}}$	(C11)
8	$O(\epsilon^{\frac{7}{3}})$ inner $[O(\epsilon^{\frac{7}{3}})$ outer fundamental$]-\Big(\frac{a_2^{(1)\pm}}{2}+\mathrm{i}\frac{\partial A}{\partial X_1}\Big)\,Y^2\epsilon^{\frac{8}{3}}$	(C12)
9	$O(\epsilon^{\frac{8}{3}})$ inner $[O(\epsilon^{\frac{8}{3}})$ outer fundamental$]$ $-\frac{2}{3}\lambda\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)\frac{1}{Y^2}\epsilon^{\frac{7}{3}}+\Big\{-\frac{61}{720}AY^6+\Big[5\mathrm{i}\lambda A-\frac{\mathrm{i}}{18}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)$ $-\frac{b_2^{(0)\pm}}{3}+\frac{\mathrm{i}}{3}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)(\frac{1}{3}\ln\epsilon+\ln\lvert Y\rvert)\Big]\,Y^3+a_3^{(1)\pm}-\Big[2\frac{\partial}{\partial T_1}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)+\frac{16\lambda}{3}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)$ $+\frac{1}{2\lambda}\Big(\frac{\partial\lvert A\rvert^2}{\partial T_1}+8\lambda\lvert A\rvert^2\Big)\,A\Big]\,[\frac{1}{3}\ln\epsilon+\ln\lvert Y\rvert]\Big\}\,\epsilon^3$	(C13)
10	$O(\epsilon^3)$ inner $[O(\epsilon^3)$ outer fundamental$]$ $+\Big\{\Big(\frac{5}{24}a_2^{(1)\pm}+\frac{\mathrm{i}}{6}\frac{\partial A}{\partial X_1}\Big)\,Y^4+\Big[2b_3^{(1)\pm}+\Big\{2\frac{\partial}{\partial X_1}\Big(\frac{\partial A}{\partial T_1}+4\lambda A\Big)-2\mathrm{i}\frac{\partial A}{\partial T_2}-2\mathrm{i}\Big(\frac{\partial a_2^{(1)\pm}}{\partial T_1}+4\lambda a_2^{(1)\pm}\Big)$ $+\Big(2b_2^{(0)\pm}+\frac{1}{\lambda}\frac{\partial\bar{P}_{2\infty}^{(0)\pm}}{\partial X_1}+\frac{1}{4\lambda}\frac{\partial\lvert A\rvert^2}{\partial X_1}+\frac{1}{\lambda}\frac{\partial b_2^{(0)\pm}}{\partial T_1}-\frac{1}{\lambda}\frac{\partial a_2^{(0)\pm}}{\partial X_1}\Big)\,A\Big\}\,\{\frac{1}{3}\ln\epsilon+\ln\lvert Y\rvert\}\Big]\,Y\Big\}\,\epsilon^{\frac{10}{3}}$	(C14)

TABLE 3

n	$O(\epsilon^{\frac{1}{3}n})$ inner $[O(\epsilon^{\frac{1}{3}n})$ outer 1st harmonic$]$	
3	0	
4	0	
5	0	
6	$-\frac{1}{4}A^2\epsilon^2$	(C 15)
7	$-\frac{1}{4}A^2\epsilon^2$	
8	$-\frac{1}{4}A^2\epsilon^2+\frac{1}{4}A^2Y^2\epsilon^{\frac{8}{3}}$	(C 16)
9	$-\frac{1}{4}A^2\epsilon^2+\frac{1}{4}A^2Y^2\epsilon^{\frac{8}{3}}+\frac{i}{8}\left(\frac{\partial A^2}{\partial T_1}+8\lambda A^2\right)\frac{1}{Y}\epsilon^{\frac{8}{3}}+a_3^{(2)\pm}\epsilon^3$	(C 17)
10	$-\frac{1}{4}A^2\epsilon^2+\frac{1}{4}A^2Y^2\epsilon^{\frac{8}{3}}+\frac{i}{8}\left(\frac{\partial A^2}{\partial T_1}+8\lambda A^2\right)\frac{1}{Y}\epsilon^{\frac{8}{3}}+a_3^{(2)\pm}\epsilon^3$ $+\left\{-\frac{1}{6}A^2Y^4+\left[3b_3^{(2)\pm}+i\left(\frac{\partial A^2}{\partial T_1}+14\lambda A^2\right)(\frac{1}{3}\ln\epsilon+\ln\lvert Y\rvert)\right]Y\right\}\epsilon^{\frac{10}{3}}$	(C 18)